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Maxime Bôcher

Geometric Aspects of the Abelian Modular Functions of Genus Four. Part II.

By Arthur B. Coble.

Introduction.

This article is a continuation of one which has appeared under the same title in this *Journal* [Vol. 46 (1924), pp. 143-192]. The sections are numbered consecutively with Part I. The notations of the earlier article are carried over without further explanation. In the meantime the author's book entitled *Algebraic Geometry and Theta Functions** has appeared. This contains an exposition of some of the matter treated in Part I along with some further results (Chap. V, **54**, **55**, **56**). It contains also a sequence of lettered theorems which are unproved [**54** (*a*), $\cdots$, (*h*); **55** (*a*), (*b*), (*c*); **56** (*a*), (*b*)]. It is the purpose of this article to develop the general subject under consideration somewhat further, and incidentally to give proofs of the theorems mentioned.

11. The Stahl Quadric of the Symmetroid and the Perspective Curves of the Planar Rational Sextic.

Let the rational sextic locus of ∞^1 planes in space $S_3(z)$, $R_2(t)$, whose point sections are apolar to the line sections of the planar rational sextic, $\bar{S}_2(t)$, be expressed parametrically (with binary parameter t) as

$$(1) \qquad (\beta z)(bt)^6 = 0.$$

Then any linear pentad on $\bar{S}_2(t)$, with points determined by $t = t_2, \cdots, t_6$ or by $(qt)^5 = 0$, which contains the further point $t = t_1$ of $\bar{S}_2(t)$, will have, for polar as to (1), the value

$$(2) \qquad (\beta z)(bq)^5(bt) = \rho \cdot (z\zeta) \cdot (t_1 t).$$

For, $(qt)^5 \cdot (t_1 t)$ is apolar to (1) for every z whence the polar of $(qt)^5$ alone is either $(t_1 t)$ or zero. The apolarity must occur for points z on a plane ζ. The locus of planes ζ, each a locus of points z with point sections apolar to a fixed linear pentad on $\bar{S}_2(t)$, is the Stahl quadric $\bar{K}$. If the linear pentad q revolves about a further point t_1 of $\bar{S}_2(t)$, i. e., if q runs over a pencil, the plane ζ of $\bar{K}$ rotates about a generator $t = t_1$ of $\bar{K}$. The

* *Colloquium Publications*, Vol. X, of the American Mathematical Society, New York, (1929); cited hereafter as "A. G."

other set of generators τ of $\bar{K}$ arises from the perspective cubic curves of $\bar{S}_2(t)$. When these are written as in **5** (1),

(3) $$(\pi x)(a\tau)(\alpha t)^3 = 0,$$

with x, ξ as dual ternary variables, the sextic $\bar{S}_2(t)$ is

(4) $$(\pi\pi'\xi)(aa')(\alpha t)^3(\alpha' t)^3 = 0.$$

The line $t = t_1$ of the perspective cubic τ in (3) meets $\bar{S}_2(t)$ in (4) in the point $t = t_1$, and in the linear pentad $(qt)^5$ given by [cf. **5** (5)],

(5) $$(\pi\pi'\pi'')(aa')(a''\tau)(\alpha t)^3(\alpha' t)^3(\alpha'' t_1)^3 = (tt_1)\cdot f(\tau^1, t_1^2, t^5) = (tt_1)\cdot(qt).^5$$

For fixed τ and variable t_1 in $f(\tau^1, t_1^2, t^5) = 0$, the plane ζ of $\bar{K}$, which corresponds to the linear pentad in t, turns about the generator τ of $\bar{K}$ [cf. A. G., **54** (24)].

If a pencil of binary sextics has an apolar net of binary quintics (rather than a pencil), the sextics are the first polars of a binary septimic to which the quintics are apolar. For given τ, the linear pentads, $f(t^5, \tau^1, t_1^2) = 0$, on the lines t_1 of the perspective cubic τ are apolar to the pencil of point sections of $R(t)$ from points z on the generator τ of $\bar{K}$. For variable t_1, these pentads lie in a net, and therefore the point sections are polars of a septimic. As τ varies, this septimic,

(6) $$(c\tau)(\gamma t)^7 = 0,$$

must lie in a pencil, and therefore must contain the parameter τ linearly, since its polars,

(7) $$(c\tau)(\gamma t_1)(\gamma t)^6 = 0,$$

must be included in the linear system (∞^3) of point sections of $R_2(t)$. We prove that

(8) *The equation* (7) *is the parametric equation* (1) *of* $\bar{R}_2(t)$ *in which the coordinates* z *in* S_3 *are referred to* $\bar{K}$ *with generators* $\tau, t_1 = t$ *as in* **7** (1) [cf. A. G., **54** (a)].

Evidently (7) is the equation of $\bar{R}_2(t)$ referred to some quadric $\bar{K}'$. Let a plane ζ of $\bar{K}'$ contain generators τ_1, t_1. Then three independent points of ζ are τ_1, t_1; τ_1, t_2; and τ_2, t_1; and the corresponding point sections of $\bar{R}_2(t)$ are $(c\tau_1)(\gamma t_1)(\gamma t)^6 = 0$, $(c\tau_1)(\gamma t_2)(\gamma t)^6 = 0$, $(c\tau_2)(\gamma t_1)(\gamma t)^6 = 0$. Since the first two are polars of $(c\tau_1)(\gamma t)^7 = 0$, their common apolar quintics are those of the net apolar to this septimic. The polars of this net as to the

septimic $(c\tau_2)(\gamma t)^7$ constitute a net of quadratics which must contain $(t_1t)^2$. Hence one quintic, $(qt)^5$, apolar to the first two sextics will have the quadratic, $(t_1t)^2$, for its polar as to $(c\tau_2)(\gamma t)^7$; and this quintic will be apolar to $(c\tau_2)(\gamma t_1)(\gamma t)^6$. Thus the three-point sections have a common apolar quintic and the plane ζ is a plane of $\bar{K}$, i. e., $\bar{K}'$ coincides with $\bar{K}$.

Let t_1, t_2 be the parameters of the double tangent of the perspective cubic determined by given τ in (3). The triad of cusps of this cubic envelope has the equation $(\bar{a}\tau)^3(\bar{a}t)^3 = 0$ (cf. **5**, pp. 163-4). Then t_1, t_2 are the roots of the hessian of this binary cubic in t, i. e.,

$$(9) \qquad (\bar{a}\,\bar{a}')^2(\bar{a}t)(\bar{a}'t)(\bar{a}\tau)^3(\bar{a}'\tau)^3 = k\cdot(tt_1)\cdot(tt_2).$$

Let the line section of $\bar{S}_2(t)$ by this double tangent consist of the four points $t = t_3, \cdots, t_6$ in addition to $t = t_1, t_2$. The three linear pentads, obtained by dropping t_1, t_2, t_3 successively from this linear hexad, determine planes on $\bar{K}$ whose parameters are, respectively, (τ, t_1), (τ, t_2), (τ', t_3), where τ' is the value of τ in $(\pi x)(a\tau)(\alpha t)^3 = 0$ when $t = t_3$ and x is $x(t_3)$ on $\bar{S}_2(t)$. The first two planes meet in the generator τ of $\bar{K}$ and the third crosses this generator at the point $z = (\tau, t_3)$ of $\bar{K}$. But these planes meet in the point $z(t_4, t_5, t_6)$ of Σ. On the other hand, the equation satisfied by a point of Σ on $\bar{K}$ is obtained by setting the catalecticant of the binary sextic (7) equal to zero, i. e.,

$$(10) \qquad (e\tau)^4(\epsilon t)^4 = 0.$$

Hence [cf. A. G., **54** (b)].

(11) *The equation of the octavic curve of intersection of Σ and $\bar{K}$ in terms of the generators τ, t on $\bar{K}$ is given by* (10). *For given τ in* (10), *the quartic in t determines the further linear tetrad on $\bar{S}_2(t)$ cut out by the double tangent of the perspective cubic τ, whose two parameters are given by* (9).

Since the double tangent of the perspective cubic envelope is itself perspective to the cubic, it is projectively related to the sextic, $\bar{S}_2(t)$. The locus of lines joining corresponding points of the double tangent and the sextic must be the perspective cubic envelope. Thus the four points, $t_3, \cdots, t_6$, on the double tangent and the sextic are self-corresponding and therefore projective. Hence [cf. A. G., **54** (c)].

(12) *The only linear tetrads of $\bar{S}_2(t)$ such that the four points on their line are projective to their parameters on $\bar{S}_2(t)$ are the sets of four residual points of $\bar{S}_2(t)$ on the double tangents of the perspective cubics of $\bar{S}_2(t)$.*

The problem of determining linear pentads of $\bar{S}_2(t)$ whose parameters on their line are projective to their parameters on $\bar{S}_2(t)$ is poristic. The projectivity imposes only two conditions on the ∞^2 possible pentads. If, however, one pentad exists which satisfies the required projective relation, the locus of lines joining corresponding points of the line and sextic, which ordinarily would be a septimic, reduces to a conic, since five points are self-corresponding. The conic is necessarily a perspective conic of the sextic and the sextic satisfies one condition. If such a perspective conic exists, the problem has ∞^1 solutions, namely, the ∞^1 tangents of the perspective conic.

The parametric equation of the double tangent, or perspective line, of a cubic envelope is of degree five in the coefficients of the parametric equation of the envelope.* Being of degree five in the coefficients of (3), the parametric equation of the double tangent has the form, $g(\xi^1, \tau^5, t^1) = 0$; and the ternary equation has the form, $g(x^1, \tau^{10}) = 0$. Hence [cf. A. G., **54** (b)].

(13) *The double tangents of the perspective cubics of $\bar{S}_2(t)$ envelop a rational curve of class* 10, $R^{10}(\tau)$. *The ten tangents of $R^{10}(\tau)$ on a point of $\bar{S}_2(t)$ divide into a set of six, and a set of four, whose parameters τ are determined by* (9) *and* (10). *For the set of six, t belongs to the pair t_1, t_2; and, for the set of four, t belongs to the tetrad $t_3, \cdots, t_6$.*

The form, $g(x^1, \tau^{10})$, is of degree ten in the coefficients of (3). When x on $\bar{S}_2(t)$ is substituted from (4), the result is the product of the forms (9) and (10). Since (9) is of degree six in the coefficients of (3), there follows that (10) has the same degree, six.

In order to interpret the form (10) on the jacobian J we recall (cf. **10**, p. 188) that, if y is a point on J, the two trihedrals of planes of the cubic curves, $C_1(\tau)$, $C_2(t)$, on y (with parameters τ_1, τ_2, τ_3; t_1, t_2, t_3 respectively) are six planes of a quadric cone, $q'(y)$, with vertex at y. If y is so placed on J that this cone breaks up into two pencils of planes on lines L, L' through y, then L must contain planes τ_1, τ_2, t_3, and L' must contain planes τ_3, t_1, t_2. Thus y is a point where an axis τ_1, τ_2 in a plane t_3 of C_2 meets an axis t_1, t_2 of C_2 in a plane τ_3 of C_1. We recall from **5** (3) that, in the plane τ_3 of C_1,

* Cf. A. B. Coble, "Symmetric Binary Forms and Involutions, III"; this *Journal*, Vol. 32 (1910), pp. 333-364; in particular, p. 353.

the perspective cubic $(\tau=\tau_3)$ is cut out by the planes of $C_2(t)$; and $\bar{S}_2(t)$ is cut out by the axes of C_1 which lie in the planes of $C_2(t)$. Hence L' is the double tangent of the perspective cubic τ_3 with parameters t_1, t_2, which meets $\bar{S}_2(t)$ at $x(t=t_3)$. This is the situation which, as observed above, gives rise to a plane $\zeta(t_3, \tau_3)$ on the Stahl quadric $\bar{K}$ tangent at the point $z(t_3, \tau_3)$ which on Σ is the point $z(t_4, t_5, t_6)$. The birationally related point on J is $y(t_4, t_5, t_6)$, the image in I [cf. **10** (2)] of the point $y(t_1, t_2, t_3)$. Hence [cf. A. G., **54** (d)].

(14) *The locus on J which corresponds to the intersection of $\bar{K}$ with Σ is the image in I of the locus of points y for which the cone $q'(y)$ of* **10** (5) *breaks up into two pencils of planes,* i. e., *the points y in which the axis of C_2 in the plane τ of C_1 meets the axis of C_1 in the plane t of C_2. For such points y, the t, τ must satisfy* (10).

A perspective envelope, E, of $\bar{S}_2(t)$ of class $m+3$ is a rational locus with parameter t such that line t of E cuts $\bar{S}_2(t)$ in the point t, and a further linear pentad which corresponds on $\bar{K}$ to a point t, τ. For every t there is just one pentad, and therefore one τ. For every τ, or perspective cubic, there are m tangents, common to the cubic and E, which have the same t since the two envelopes generate $\bar{S}_2(t)$ of order six. Hence the parameters t, τ of the pentads of E satisfy a relation of the form, $(k\tau)(\kappa t)^m=0$. The parametric equation of E is obtained by the elimination of τ from this relation and $(\pi x)(a\tau)(\alpha t)^3=0$. Hence [cf. A. G., **54** (e)].

(15) *The family of perspective envelopes of $\bar{S}_2(t)$ of class $m+3$ corresponds in* $(1,1)$ *fashion to the family of ∞^{2m+1} curves on $\bar{K}$ of order $m+1$ and type $(k\tau)(\kappa t)^m=0$. The parametric equation of the family is $(\pi x)(ak)(\kappa t)^m(\alpha t)^3=0$.*

In this last theorem it has seemed more convenient to think of t, τ as a point on $\bar{K}$. We shall continue this view throughout the following discussion of the perspective quartic envelopes of $\bar{S}_2(t)$. The symmetroid Σ is then the quartic envelope of planes which cut a rational space sextic curve, $R_2(t)$, in catalectic sextics; and $\bar{K}$ is the locus of points z such that plane sections of $\bar{R}_2(t)$ by planes on z have a common apolar quintic which determines a linear pentad on $\bar{S}_2(t)$. If $(m\tau)(\mu t)=0$ is the equation of a generic plane section of $\bar{K}$, whose coordinates ζ are the coefficients $m_{ij}=m_i\mu_j$ [cf. **7** (2)], then

(16) $$(\pi x)(am)(\mu t)(\alpha t)^3=0$$

is the generic perspective quartic envelope of $\bar{S}_2(t)$. If this is written in the form,

(17) $$\xi_0 = (\beta t)^4,\ \xi_1 = (\gamma t)^4,\ \xi_2 = (\delta t)^4,$$

where the coefficients β, γ, δ are linear in m_{ij}, the condition that the perspective envelope have a perspective line is (cf. *loc. cit.*)

(18) $$\begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & 0 \\ 0 & \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & 0 \\ 0 & \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_0 & \delta_1 & \delta_2 & \delta_3 & \delta_4 & 0 \\ 0 & \delta_0 & \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{vmatrix} = 0.$$

This condition, of degree 6 in $\zeta = m_{ij}$, is the product of the equations of the quadric envelope $\bar{K}$, and of the quartic envelope Σ. For, the curve (16) has a perspective line, either if it is a proper quartic envelope which has a tritangent, or if it is a perspective cubic with a bitangent. The second case occurs when $(m\tau)(\mu t)$ factors into $(t_1 t)\cdot(\tau_1\tau)$. Then (16) is the perspective cubic τ_1, i. e., $(\pi x)(a\tau_1)(\alpha t)^3\cdot(t_1 t) = 0$; also the plane $\zeta = m_{ij}$ is a plane of $\bar{K}$. If, on the other hand, (16) is a proper quartic envelope with a tritangent ξ with parameters t_1, t_2, t_3 then ξ cuts $\bar{S}_2(t)$ in points $t_1, \cdots, t_6$, and the three pentads, obtained by dropping t_1, t_2, t_3 successively, correspond to three points on $\bar{K}$, and on its section by the plane $\zeta = m_{ij}$ which is also the plane $\zeta(t_4, t_5, t_6)$ of Σ. Hence

(19) *The tritangent condition of the curve* (16) *of class four factors into the equation,* $(mm')(\mu\mu') = 0$, *of* $\bar{K}$, *and into the equation of the symmetroid* Σ *of class four, and degree six in the coefficients of* (3).

If, in this equation of Σ, the m, μ are replaced by t, τ, the equation (10) of the section of Σ by $\bar{K}$ is obtained.

If $x' = x'(t')$ is a point of $\bar{S}_2(t)$ with parameter t', then

$$(\pi x')(a\tau)(\alpha t)^3 = (tt')\cdot f(\tau^1, t^2),$$

since, for any τ, one of the three tangents on x' of the perspective cubic has the parameter t'. If p_i $(i = 1, \cdots, 9, 0)$ is one of the ten nodes, $P_{10}{}^2$, of $\bar{S}_2(t)$, with nodal parameters $(q_i t)^2$, then

(20) $$(\pi p_i)(a\tau)(\alpha t)^3 = (\lambda_i t)(l_i\tau)\cdot(q_i t)^2.$$

Now the pencil of lines on p_i, covered four times, is a degenerate perspective quartic of $\bar{S}_2(t)$. This particular perspective curve is obtained when

$(m\tau)(\mu t)$ in (16) is $(\lambda_i t)(l_i \tau)$. For, if τ is eliminated between (20) and (3), the resulting line,

$$(\pi x)(\pi' p_i)(a a')(\alpha t)^3(\alpha' t)^3 = 0,$$

vanishes when $x = p_i$ for every t, since the left member then changes sign on interchange of equivalent symbols. Let the "incidence condition" of the perspective quartic be the condition that its tangents t_1, t_2, t_3 are on a point. Formed for (16), it has the degrees,

(21) $$f(m^3, \mu^3, t_1^2, t_2^2, t_3^2) = 0.$$

The tritangent condition of (19) is of degree two in the coefficients of (21). But when the perspective curve is the four-fold point p_i, the condition (21) is satisfied for all values of t_1, t_2, t_3. Hence, for variation of t_1, t_2, t_3, the envelope (21) is in a linear system (∞^9) of cubic envelopes on the 10 planes, $(\lambda_i t)(l_i \tau) = 0$, of (20). Since, according to (19), $\Sigma\, \bar{K}$ is a quadratic in these cubic envelopes, the 10 planes must be the double planes, or tropes, of Σ. Hence

(22) *The ten bilinear equations,* $(\lambda_i t)(l_i \tau) = 0$, *in* (20) *are the sections of* $\bar{K}$ *by the ten tropes of* Σ. *If* A_i *is a common axis on* J *of the cubic curves,* $C_1(\tau)$, $C_2(t)$, *and if* A_i *is on the two planes* $(q_i \tau)^2$ *of* C_1 *and the two planes* $(q_i t)^2$ *of* C_2, *then, as* y *runs over* A_i, *the further planes* t, τ *of* C_1, C_2 *on* y *are related as in* $(\lambda_i t)(l_i \tau) = 0$. *For variation of* t_1, t_2, t_3 *in* (21), *the linear system of cubic adjoint surfaces of* Σ *is obtained.*

The statement concerning the axes A_i of J is an immediate consequence of the constructions given in **5** (3).

12. Loci Associated with a Division of the Nodes of the Rational Sextic and the Symmetroid.

Let $q(i, t)^0$ $(i = 1, \cdots, 9, 0)$ denote the binary quadratic in t which determines the pair of nodal parameters of $\bar{S}_2(t)$ at the node p_i; and $q(ij, t)^1$ the quadratic which determines the parameters of the two further intersections of $\bar{S}_2(t)$ with a line on the nodes $p_i p_j$. The corresponding quadratics for the paired sextic $\bar{S}_1(\tau)$ will be distinguished by the parameter τ.

The pencil of lines on the node p_1 of $\bar{S}_2(t)$ will cut $\bar{S}_2(t)$ in the fixed pair of points $q(1, t)^0$ and a variable tetrad t_3, t_4, t_5, t_6 which lies in an I_1^4 with equation

(1) $$f(t_3^3, t_4^3) = 0.$$

The planes $t_3, \cdots, t_6$ of the variable tetrad in $I_1{}^4$ on the cubic curve $C_2(t)$ will determine a tetrahedron whose vertices will run over a twisted cubic curve N_1. To the particular vertex $y(t_4, t_5, t_6)$ of N_1 on J, opposite to the plane t_3, we may ascribe the binary parameter $t = t_3$; and thus there is established on N_1 a binary coordinate system $N_1(t)$. Then the axis t_3, t_4 of $C_2(t)$ is the bisecant t_5, t_6 of $N_1(t)$. Two cubic curves so placed that an infinity of bisecants of the one are axes of the other are called Hurwitz curves (cf. Meyer, Apolarität, p. 315). The particular line on p_1 which passes through p_2 cuts $\bar{S}_2(t)$ in the tetrad $q(2,t)^0 \cdot q(12,t)^1$. Hence the axis A_2, or $q(2,t)^0$, of $C_2(t)$, one of the ten common axes of $C_1(\tau)$, $C_2(t)$, is the bisecant $q(12,t)^1$ of $N_1(t)$. The point $t = t_3$ of $N_1(t)$ is the point $y(t_4, t_5, t_6)$ on J; and this is the image in the involution I of the point $y[q(1,t)^0, t_3]$ of A_1 on J.

In precisely the same way the curve N_1 arises from the $I_1{}^4$ cut out on $\bar{S}_1(\tau)$ by lines on its node q_1. The axis A_2 is the axis $q(2,\tau)^0$ of $C_1(\tau)$ as well as the bisecant $q(12,\tau)^1$ of N_1. The parameter τ on N_1 is that of the point of A_1 which corresponds under I to the point of N_1 whence the parameters t, τ on N_1 are related by the bilinear equation, $(\lambda_1 t)(l_1 \tau) = 0$, of **11** (22).

Since $q(12,t)^1$ and $q(12,\tau)^1$ are interchanged under the projectivity $(\lambda_1 t)(l_1 \tau) = 0$ and therefore also under the projectivity, $(\lambda_2 t)(l_2 \tau) = 0$, either quadratic must be fixed under the product of the projectivities. Hence [cf. A. G., **54** (f), (g)].

(2) *The cubic curve N_i on J, the transform of the line A_i on J under I, has the common axis A_j of $C_1(\tau)$, $C_2(t)$ for bisecant with parameters $q(ij,\tau)^1$ or $q(ij,t)^1$ $(i, j = 1, \cdots, 9, 0; i \neq j)$. The parameters t, τ on N_i are related by $(\lambda_i t)(l_i \tau) = 0$. The 45 pairs of quadratics $q(ij,t)^1, q(ij,\tau)^1$ arise from the common solutions of the 45 pairs of projectivities, $(\lambda_i t)(l_i \tau) = 0$, $(\lambda_j t)(l_j \tau) = 0$, i. e., $q(ij,t)^1 = (\lambda_i t)(\lambda_j t)(l_i l_j)$, and $q(ij,\tau)^1 = (\lambda_i \lambda_j)(l_i \tau)(l_j \tau)$.*

We have seen [cf. **10** (14)] that the points of an axis A_i on J correspond to the directions at the node D_i of Σ. The curve N_i is the image of A_i in the involution I. This involution on J can be effected by the cubic Cremona involution determined by any net of quadrics in the web whose jacobian is J. The Cremona involution has a sextic F-curve which corresponds to a plane section of Σ. Since the involution transforms N_i back into A_i, N_i must meet the F-curve in 8 points. Hence N_i must correspond on Σ to a rational octavic $m_i{}^8$. Since N_i is cut by A_j in a pair of points with parameters $q(ij,t)^1$, the octavic $m_i{}^8$ must have a node at D_j with nodal parameters

$q(ij,t)^1$. The quadric M_i on the nine nodes other than D_i meets m_i^8 in 18 points at the nodes, and therefore contains it, i. e., m_i^8 is the intersection of Σ and M_i [cf. A. G., **54** (h)].

The equation satisfies by planes t, τ of the Stahl quadric $\bar{K}$ on the node D_i, or also the equation of the conic section of $\bar{K}$ by the polar plane of D_i, is $(\lambda_i t)(l_i \tau) = 0$ [cf. **11** (2) in dual form]. Hence the 45 quadratics in (2) are the parameters t (or parameters τ) of the pairs of planes of $\bar{K}$ on each of the 45 lines joining a pair of nodes D_i, D_j.

The following theorem is useful as a lemma [cf. A. G., **55** (a)]:

(3) *If two generic cubic space curves N_1, N_2, and eight of their ten common bisecants, say $A_3, \cdots, A_9, A_0$ are given, there exists a unique quartic surface J which contains N_1, N_2, and A_j $(j = 3, \cdots, 9, 0)$. This surface J is the jacobian of a web of quadrics apolar to the quadric envelopes on a unique pair of cubic curves C_1, C_2 for which the A_j are common axes. Either curve C with either curve N is a pair of Hurwitz curves. The relation between the pairs N_1, N_2 and C_1, C_2 is mutual but the transition from one pair to the other is reversed by a dual process.*

The equation of a quartic surface J contains 35 coefficients. To contain N_1, N_2 imposes 26 linear conditions upon these coefficients, and to contain the eight A_j imposes 8 more. There exists therefore at least one surface J on these curves. If there were a pencil of surfaces J, the base curve of the pencil would consist of N_1, N_2, A_j and a further conic B. This conic B could not meet every A_j in two points since the A_j are not coplanar. Suppose it meets A_3 in at most one point. Pass a plane π through A_3 to avoid the points common to two base curves which are not on A_3. Then π would meet the pencil in A_3, and in a pencil of cubic curves on a further point of N_1, of N_2, of $A_4, \cdots, A_9, A_0$, and of B. This pencil on ten points would have a common part which would belong to the base curve. But every plane section on A_3 could not contain an additional part of the base curve and thus J must be unique. The two cubic curves N_1, N_2 determine two paired planar rational sextics for which on either the eight common bisecants determine eight of the nodes. On N_1 (and similarly on N_2) the two further nodes determine two I_1^4's for which the eight pairs of parameters of A_j are pairs of the I_1^4. The planes of the triads of these I_1^4's envelope curves $C_1(\tau), C_2(t)$ for which the A_j are common axes. The jacobian J of the web of point quadrics apolar to the nets of quadric envelopes on $C_1(\tau), C_2(t)$ must contain the 8 common axes A_j of $C_1(\tau), C_2(t)$; and therefore must contain N_1, N_2, and must coincide

with the J uniquely determined as above. Thus $C_1(\tau), C_2(t)$ are related to N_1, N_2 in a manner dual to that in which N_1, N_2 are related to $C_1(\tau), C_2(t)$. From this there follows [cf. A. G., **55** (b)]:

(4) *A planar rational sextic curve is uniquely determined, to within a projectivity, when the pairs of parameters of eight nodes are given.*

Naturally the eight pairs of parameters can not be selected at random. We take up later (cf. **14**) the relations among the further nodal pairs of the five rational sextics which are projectively determined when six nodal pairs are given at random. This theorem (4) is the basis for the proof (cf. A. G., p. 251) that the rational sextic, (N_1, N_2), whose nodal pairs are those cut out on N_1 by the ten common bisecants of N_1, N_2, is the transform of $\bar{S}_2(t)$, or $(\bar{C}_2, \bar{C}_1)$, by $J^5{}_{1,2}$, the Jonquières transformation of order five with fourfold F-point at the node p_1 of $\bar{S}_2(t)$ and simple F-points at $p_3, \cdots, p_9, p_0$. From this in turn there is derived a Cremona transformation, R_1, which will transform $\bar{S}_2(t)$ into its paired sextic $\bar{S}_1(\tau)$, i. e., will transform $(\bar{C}_2, \bar{C}_1)$ into $(\bar{C}_1, \bar{C}_2)$.

The symmetroid Σ with nodes at $D_1, \cdots, D_0$ is transformed by the cubic transformation A_{1234} with F-points at $D_1, \cdots, D_4$ into a symmetroid Σ' with nodes at $D_1', \cdots, D_0'$ [cf. A. G., **53** (7)]. We seek a Cremona transformation which will transform the J of Σ into the J' of Σ'. Such a transformation is not unique since J is transformed into itself by an infinite Cremona group. We prove that a simple transformation of the type required is the cubic transformation with F-curves consisting of N_1 and its bisecants A_2, A_3, A_4.

The generic spatial cubic transformation has sextic curves of genus three for direct and inverse F-curves such that the P-surface of either is the octavic locus of trisecants of the other. In the particular case when the one F-curve in $S_3(y)$ degenerates into a cubic curve c^3 and three bisecants b_1, b_2, b_3 the other in $S_3(Y)$ degenerates into a cubic curve C^3 and three bisecants B_1, B_2, B_3. The P-surface in $S_3(y)$ consists of four reguli r, r_1, r_2, r_3 whose lines correspond respectively to the points on C^3, B_1, B_2, B_3. The regulus r crosses b_1, b_2, b_3, and the regulus r_i crosses c^3, b_j, b_k $(i, j, k = 1, 2, 3)$. The P-surface in $S_3(Y)$ with reguli R, R_1, R_2, R_3 is similarly defined with reference to C^3, B_1, B_2, B_3. From the image in $S_3(Y)$ of a plane on b_i in $S_3(Y)$ there factors the regulus R_i leaving a plane on B_i. Thus the pencils of corresponding planes on b_i, B_i are projectively related.

The image of a generic line in $S_3(y)$ is a cubic curve bisecant to C^3, B_1, B_2, B_3. If the line is a bisecant of c^3, two rays of R separate from

the image leaving a line bisecant to C^3. Thus the transformation sets up a correspondence between the bisecants of c^3 and the bisecants of C^3 which has been studied by Montesano (cf. A. G., p. 258 for references). If the parameters t, t' of the points of intersection of a bisecant of c^3 are used to determine a point p in a plane with reference to a norm conic, the bisecants of c^3 are mapped on the points p of a plane. With a similar mapping for the bisecants of C^3, the correspondence mentioned turns out to be quadratic with F-triangles determined by b_1, b_2, b_3 and B_1, B_2, B_3.

The image of a quartic surface on c^3, b_1, b_2, b_3 is a quartic surface on C^3, B_1, B_2, B_3. The jacobian J of a web of quadrics can be defined to be a quartic surface containing N_1, N_2 and the eight common bisecants $A_3, \cdots, A_9, A_0$. If N_1, A_3, A_4, A_5 are identified with c^3, b_1, b_2, b_3 the image of N_2 is a cubic curve N_2' bisecant to B_1, B_2, B_3. The images of $A_6, \cdots, A_9, A_0$ are common bisecants, $A_6', \cdots, A_9', A_0$ of N_2' and C^3. If C^3, B_1, B_2, B_3 are denoted by N_1', A_3', A_4', A_5' the image J' of J is a quartic surface on N_1', N_2', and eight of their common bisecants, $A_3', \cdots, A_0'$. Hence

(5) *The jacobian J of a web of quadrics is transformed by the cubic transformation whose sextic F-curve consists of N_i, A_j, A_k, A_l into a jacobian J'.*

The jacobian J is defined by the axes A_i, or by the sextic curve $\bar{S}_2(t) = (\bar{C}_2(t), \bar{C}_1(\tau))$. We seek the sextic curve which defines J'. The sextic curve (N_1, N_2) with eight common bisecants $A_3, \cdots, A_0$ is the transform of $\bar{S}_2(t)$ by $J^5{}_{1,2}$. Under the cubic transformation with sextic F-curve, N_1, A_3, A_4, A_5, the effect on the bisecants is obtained by first plotting their pairs of parameters with respect to a norm conic. This plotting operation on the bisecants which determine the nodes of (N_1, N_2) furnishes in the plane the nodes of the sextic paired with (N_1, N_2) [cf. **4** (4)]. But in the plane of the sextic the paired sextic is obtained by effecting the transformation R_1 (cf. A. G., p. 252). In the plane of the norm conic the quadratic transformation is then applied to obtain the pairs of parameters of the new set of nodes with respect to the new norm conic. These pairs are restored to the cubic curve N_1' by a second use of R_1. On N_1' the nodal set of (N_1', N_2') determines the lines A_i' of J'. These are axes for the sextic $\bar{S}_2'(t) = (\bar{C}_2'(t), \bar{C}_1'(\tau))$ which arises from the sextic (N_1', N_2') by a second application of $J^5{}_{1,2}$. Hence the sextic $\bar{S}_2'(t)$ which defines J' is the transform of the sextic $\bar{S}_2(t)$ which defines J by the Cremona transformation

(6) $$T = J^5{}_{1,2} R_1 A_{345} R_1 J^5{}_{1,2}.$$

In order to prove that T is the quintic transformation A_{678902} with double F-points at nodes $p_2, p_6, \cdots, p_9, p_0$ we observe first that $J^5_{2,1} = (12)J^5_{1,2}(12)$, i. e., $J^5_{2,1}$ is the transform of $J^5_{1,2}$ by the transposition (12). Since $R_1 = J^5_{1,2}J^5_{2,1}(12)$ (cf. A. G., p. 252), we have, on eliminating $J^5_{2,1}$, that $R_1 = J^5_{1,2}(12)J^5_{1,2}$, whence

$$T = (12)J^5_{1,2}A_{345}J^5_{1,2}(12).$$

Furthermore

$$J^5_{1,2} = (34)(56)(78)(90)A_{190}A_{178}A_{156}A_{134} \\ = A_{134}A_{156}A_{178}A_{190}(90)(78)(56)(34).$$

Hence

$$T = (12)(34)(56)(78)(90)A_{190}A_{178}A_{156}A_{134} \cdot A_{345} \cdot \\ A_{134}A_{156}A_{178}A_{190}(90)(78)(56)(34)(12).$$

The following transforms are readily checked:

$$A_{134}A_{345}A_{134} = (15)\,;\ A_{156}(15)A_{156} = (15)\,;$$
$$A_{178}(15)A_{178} = A_{578}\,;\ A_{190}A_{578}A_{190} = A_{157890}\,;$$
$$(12)(34)(56)(78)(90)A_{157890}(90)(78)(56)(34)(12) = A_{267890}.$$

When the rational sextic is transformed into a congruent sextic by the quintic transformation, A_{267890}, the symmetroid of the one is transformed into the symmetroid of the other by the regular cubic transformation A_{1345}. Hence [cf. A. G., **55** (c)].

(7) *If J, Σ and the pair of rational sextics, $\bar{S}_2(t)$, $\bar{S}_1(\tau)$, a birationally related triad, are transformed by Cremona transformation into a similarly related J', Σ' and pair $\bar{S}_2'(t)$, $\bar{S}_1'(\tau)$, the transformations are such that the cubic transformation with sextic F-curve, N_1, A_2, A_3, A_4, on J, the regular cubic transformation with F-points at nodes $D_1, \cdots, D_4$ on Σ, and the quintic transformation $A_{5,\ldots,9,0}$ with double F-points at the nodes $p_5, \cdots, p_9, p_0$ of the sextic, correspond.*

By taking products of corresponding transformations of the types given in (7), more general corresponding transformations are obtained.

13. Linear Systems of Curves on J and Σ Referred to a Base.

A brief account of the Severi theory of a base for linear systems of curves on an algebraic surface is given by its author in Pascal's Repertorium.* This

* II_2, pp. 763-6; Teubner, Leipzig (1922).

theory is applied by Snyder and Sharpe * to a discussion of certain involutions defined on J and Σ. We use their results to discuss some of the foregoing transformations.

In conformity with the notation introduced earlier (cf. **10**), let η denote the linear system of plane sections of J; B_6 (the $C_3'^6$ of **10**), the transform of η under I; C_6, the linear system of loci of nodes of nets of quadrics contained within the web which defines J; A_i $(i = 1, \cdots, 10)$, one of the ten lines on J; and N_i, the transform of A_i under I. The Snyder-Sharpe notation for these is $C_4, C_6, \bar{C}_6, \gamma_i, (\gamma_i)$, respectively. They show that η, B_6, and nine of the ten lines A_i constitute a minimum base for the surface and that the tenth line A is related to this base as in

$$(1) \qquad 7\eta \equiv 3B_6 + \Sigma_i A_i.$$

On account of the symmetry it is convenient either to retain all twelve of the curves, or to drop B_6. In the latter case the η, and the ten curves A_i, constitute a base, but not a minimum base, since B_6 can not be expressed integrally in terms of them.

If $[C_i, C_k]$ denotes the number of common points of the curves C_i, C_k; and if $[C_i, C_i]$ denotes the virtual grade of the linear system C_i, then (cf. S. S., pp. 287-8).

$$(2) \qquad \begin{aligned} &[\eta, \eta] = 4,\ [\eta, B_6] = 6,\ [\eta, A_i] = 1, \\ &[B_6, B_6] = 4,\ [B_6, A_i] = 3,\ [A_i, A_i] = -2,\ [A_i, A_j] = 0. \end{aligned}$$

If two curves, C, C' are expressed in terms of the base by the relations,

$$(3) \qquad C = \Sigma_i \lambda_i C_i, \quad C' = \Sigma_i \lambda_i' C_i,$$

the number of intersections of the two is given by the formula

$$(4) \qquad \Sigma_{i,k} \lambda_i \lambda_k' \,[C_i, C_k].$$

This is the polar of the *fundamental quadratic form* of the surface

$$(5) \qquad \Sigma_{ik} \lambda_i \lambda_k \,[C_i, C_k].$$

Cubic surfaces on a curve B_6 cut out the linear system C_6 [cf. **10** (17)], whence $B_6 + C_6 = 3\eta$. We then have from (2) and (4) that

$$(6) \qquad [C_6, \eta] = 6,\ [C_6, C_6] = 4,\ [C_6, A_i] = 0.$$

It is convenient to use for a base on J the 11 curves C_6, A_i. The plane sections are connected with these by the relation

$$(7) \qquad 2\eta = 3C_6 - \Sigma_i A_i.$$

* "Space involutions defined by a web of quadrics," *Transactions of the American Mathematical Society*, Vol. 19 (1918), pp. 275-290; cited hereafter as S. S.

With respective multiples λ_0 and λ_i $(i=1,\cdots,10)$ for these curves the fundamental quadratic form for J is

$$(8) \qquad 4\lambda_0^2-2\lambda_1^2-\cdots-2\lambda_{10}^2.$$

In passing to the birationally related symmetroid (cf. S. S., p. 290) C_6 on J becomes a plane section ζ of Σ, the lines A_i on J become the nodes D_i of Σ, and the quadric section 2η of J becomes the section of Σ by an adjoint cubic surface [cf. **10** (15)]. Thus the base finally retained for J becomes, on Σ, the base ζ, D_i.

If the surface J (or Σ) is birationally transformed into itself, the basal curves C_i are transformed into basal curves C_i' which in turn can be expressed in terms of the C_i. The linear transformation which thus arises must leave the fundamental quadratic form unaltered. Since, according to (3), the λ's behave contragrediently to the curves C this linear transformation on the curves C will have, as an invariant quadratic form, the dual of (8), i. e.,

$$(9) \qquad \begin{aligned} &C_6^2-2A_1^2-\cdots-2A_{10}^2 \text{ (on } J);\\ &\zeta^2-2D_1^2-\cdots-2D_{10}^2 \text{ (on } \Sigma). \end{aligned}$$

Furthermore if J (or Σ) is birationally transformed into a projectively similar, though not a projectively identical, surface, a like linear transformation appears which has the same quadratic invariant. If for example Σ is transformed into a symmetroid Σ' by the regular cubic transformation A_{1234} with F-points at $D_1,\cdots,D_4$, the plane section ζ' of Σ' arises from the section, $3\zeta-2D_1-2D_2-2D_3-2D_4$, of Σ by the cubic surface with nodes at $D_1,\cdots,D_4$; and the node D_1' of Σ' arises from the quartic curve, $\zeta-D_2-D_3-D_4$, in which Σ is cut by the plane on D_2, D_3, D_4. The linear transformation then has the form

$$(10) \qquad \begin{aligned} &\zeta'\equiv\zeta+2\pi_{1234} && (\pi_{1234}=\zeta-D_1-\cdots-D_4),\\ &D_i'\equiv D_i+\pi_{1234} && (i=1,\cdots,4),\\ &D_j'\equiv D_j && (j=5,\cdots,10). \end{aligned}$$

But in this transformation to a surface with merely similar basal curves there is no reason why the order of the nodes of the one should have in advance any relation to the order of the nodes of the other so that any permutation π of the nodes of either might be adjoined in representing the transformation. By comparison with A. G. **15** it is clear that the linear group on the basal curves

thus induced by regular transformation of Σ into Σ', and by permutation of the nodes, is the transposed form of the group $g_{10,3}$ determined by the congruent nodal sets of Σ, Σ'.

In terms of the base ζ, D_i on Σ, or C_6, A_i on J, certain involutions on

Σ, J take somewhat simpler forms than those given by Snyder-Sharpe (cf. S. S., pp. 288-9). The involution I of pairs on J apolar to the web is

$$(11)\qquad \begin{array}{ll} \zeta' \equiv \zeta + 4\pi_{1,\ldots,10} & (\pi_{1,\ldots,10} = 2\zeta - D_1 - \cdots - D_{10}), \\ D_i' \equiv D_i + 2\pi_{1,\ldots,10} & (i = 1, \cdots, 10). \end{array}$$

This is permutable with (10) whence if Σ is transformed into Σ' by regular Cremona transformation the pairs of I on Σ are transformed into the pairs of I' on Σ'.

The involution on Σ obtained by drawing lines through the node D_1, which on J is the involution whose pairs are bisecants of N_1, is

$$(12)\qquad \zeta' \equiv 3\zeta - 4D_1,\ D_1' \equiv 2\zeta - 3D_1,\ D_j' \equiv D_j\ (j = 2, \cdots, 10).$$

We determine finally the expression for the transformation of J by the cubic Cremona transformation whose sextic F-curve is N_1, A_2, A_3, A_4 [cf. **12** (7)]. The plane sections η' arise from the residual intersection of J with cubic surfaces on N_1, A_2, A_3, A_4, i. e., $\eta' \equiv 3\eta - N_1 - A_2 - A_3 - A_4$. Since $N_1 \equiv B_6 - \eta + A_1$ (cf. S. S.), $\eta' \equiv 4\eta - B_6 - A_1 - A_2 - A_3 - A_4$. The curve N_1' on J' arises from the residual intersection with J of the regulus on A_2, A_3, A_4 whence $B_6' - \eta_1' + A_1' \equiv 2\eta - A_2 - A_3 - A_4$. The line A_j' on J' arises from the residual intersection with J of the regulus on A_k, A_l, N_1 $(j, k, l = 2, 3, 4)$, whence $A_j' \equiv 2\eta - N_1 - A_k - A_l \equiv 3\eta - B_6 - A_1 - A_k - A_l$. The line A_i' on J' arises from the line A_i on J whence $A_i' \equiv A_i$ $(i = 5, \cdots, 10)$. The identity

$$7\eta' - 3B_6 - \Sigma_i A_i \equiv 0 \equiv 7\eta' - 3B_6' - \Sigma_i A_i'$$

then suffices to determine A_1'. On eliminating η and B_6 in favor of C_6, we have

$$(13)\qquad \begin{array}{ll} C_6' \equiv C_6 + 2\rho_{1234} & (\rho_{1234} = 2C_6 - A_1 - \cdots - A_4), \\ A_i' \equiv A_i + \rho_{1234} & (i = 1, \cdots, 4), \\ A_j' \equiv A_j & (j = 5, \cdots, 10). \end{array}$$

Since this is the same transformation on J as (10) on Σ the theorem [**12** (7)] is confirmed.

Furthermore, if Σ is transformed into Σ' by A_{1234} and Σ' into Σ'' by A'_{1234}, then Σ and Σ' are projective, and there exists a collineation C such that the product $A_{1234}A'_{1234}C$ is the identical collineation. Hence if J is transformed into J' by the Cremona transformation $T_{1,234}$ with F-curves N_1, A_2, A_3, A_4, and if J' is similarly transformed into J'' by $T_{2,134}$ there must exist a collineation C' which transforms J'' into J. Then the product $T_{1,234}T_{2,134}C'$ is a Cremona transformation of order five which on J is the

identity. This order five is probably the lowest for a transformation having J as a locus of fixed points.

14. Projective Configurations Defined by Nodal Parameters of the Rational Sextic.

It is known (cf. Meyer, Apolarität, p. 317) that there are five projectively distinct planar rational sextics with six given pairs of nodal parameters; and that, if one of the five is selected, the remaining four are obtained from it by quadratic transformation with F-points at any three of the four remaining nodes.

Let the nodal parameters of $\bar{S}_2(t)$ at the nodes $p_5, \cdots, p_9, p_{10}$ be regarded as known, and let $r_{01}, r_{02}, r_{03}, r_{04}$ denote the binary quadratics which determine the nodal parameters at the nodes $p_1, \cdots, p_4$; furthermore, let r_{ij} denote the quadratic which determines the parameters of the two further points of $\bar{S}_2(t)$ on the line joining the nodes p_k, p_l $(i, j, k, l = 1, \cdots, 4)$. If $p_1, \cdots, p_4$ are taken in order to be the reference points and unit point, then the parametric equation of $\bar{S}_2(t)$ is

$$(1) \qquad \begin{array}{ll} x_1 = r_{02}r_{03}r_{14}, & x_2 - x_3 = r_{04}r_{01}r_{23}, \\ x_2 = r_{03}r_{01}r_{24}, & x_3 - x_1 = r_{04}r_{02}r_{31}, \\ x_3 = r_{01}r_{02}r_{34}, & x_1 - x_2 = r_{04}r_{03}r_{12} \\ & (r_{ij} = -r_{ji}). \end{array}$$

Thus the four pairs of nodal parameters, in addition to the six given pairs, of these five rational sextics are

$$(2) \qquad \begin{array}{rlllll} \bar{S}_2(t) = S^{(0)}: & & r_{01}, & r_{02}, & r_{03}, & r_{04}; \\ S^{(1)}: & r_{10}, & & r_{12}, & r_{13}, & r_{14}; \\ S^{(2)}: & r_{20}, & r_{21}, & & r_{23}, & r_{24}; \\ S^{(3)}: & r_{30}, & r_{31}, & r_{32}, & & r_{34}; \\ S^{(4)}: & r_{40}, & r_{41}, & r_{42}, & r_{43}, & \end{array}$$

where the sextic $S^{(i)}$ arises from $S^{(0)}$ by the quadratic transformation with F-points at p_j, p_k, p_l.

The 10 quadratics r_{ij}, either in conjunction with the six given nodal pairs or by themselves define interesting projective configurations which we proceed to discuss. With reference to a norm-conic $\bar{K}(t)$ we plot the points r_{ij} determined by the ten quadratics r_{ij}; and denote by R_6^2 the six points similarly plotted from the six given pairs of nodal parameters common to the five rational sextics $S^{(i)}$ in (2). These five sextics constitute a symmetrical set in that the four nodes of $S^{(i)}$ have nodal parameters r_{ij} and that the further pair of points on the line joining nodes r_{ij}, r_{ik} has parameters r_{lm} $(i, \cdots, m = 0, 1, \cdots, 4)$. According to **4** (4), the six points R_6^2 and the four points r_{ij} $(j \neq i)$ constitute the nodes of a sextic $S^{(i)}(\tau)$ paired

with the sextic $S^{(4)}(t)$ in (2). According to A. G., **55** (7), the sextic $S^{(4)}(\tau)$ arises from the sextic $S^{(j)}(\tau)$ by the Geiser involution with seven F-points at R_6^2 and r_{ij}. The pairs of points, r_{ik}, r_{jk}; r_{il}, r_{jl}; r_{im}, r_{jm} are pairs of this involution. Each pair together with the seven F-points are the base points of a pencil of cubics. The particular cubic $C^{(m)}$ common to the first two pencils contains $r_{ij}, r_{ik}, r_{jk}, r_{il}, r_{jl}$ and it must also contain r_{kl}. For, it is determined by the fact that it contains $r_{ik}, r_{jk}, r_{il}, r_{jl}$. Thus we find five cubics $C^{(m)}$ on R_6^2 such that $C^{(l)}, C^{(m)}$ meet in r_{ij}, r_{ik}, r_{jk}.

If the plane of $\bar{K}(t)$ is mapped upon a cubic surface Γ^3 by means of cubic curves on R_6^2, the five cubics $C^{(m)}$ give rise to an inscribed five-plane of Γ^3 whose ten vertices on Γ^3 are the images of the ten points r_{ij}. The five sextics $S^{(j)}(\tau)$ give rise to sections of Γ^3 by quadrics which touch Γ^3 at the vertices of one of the five tetrahedra contained in the inscribed five-plane.

Conversely, let a cubic surface Γ^3 contain an inscribed five-plane with planes $y_0, \cdots, y_4$ such that

(3) $$y_0 + y_1 + y_2 + y_3 + y_4 = 0.$$

The equation of Γ^3 is then

(4) $$\Gamma^3 = \Sigma a_{ijk} y_i y_j y_k = 0.$$

The tetrahedron, $y_0 = y_1 = y_2 = y_3 = 0$, has vertices

$$1, 0, 0, 0, -1;\ 0, 1, 0, 0, -1;\ \cdots\cdots;\ 0, 0, 0, 1, -1.$$

The tangent plane of Γ^3 at one of these four points is

$$a_{014} y_1 + a_{024} y_2 + a_{034} y_3 = 0.$$

The quadric,

(5) $$Q^{(4)} = a_{014} y_0 y_1 + a_{024} y_0 y_2 + \cdots + a_{234} y_2 y_3 = 0,$$

passes through these four points and has, at these points, the same tangent planes as Γ^3. One such quadric determines the entire set of five, $Q^{(4)}$. For, if $Q^{(4)}$ touches Γ^3 at the vertices of a tetrahedron, a plane y_4 can be determined in such wise that $\Gamma^3 - y_4 Q^{(4)}$ is four-nodal at the four vertices and therefore has the form

$$a_{012} y_0 y_1 y_2 + \cdots + a_{123} y_1 y_2 y_3 = 0.$$

Hence if a quadric touches Γ^3 at four vertices of a tetrahedron, the planes of the tetrahedron are part of an inscribed five-plane. But these planes are part of such a five-plane if the six edges of the tetrahedron meet Γ^3 again in six points on a plane. Hence [cf. A. G., **56** (a)].

(6) *Ten points $p_1, \cdots, p_4$; $p_5, \cdots, p_{10}$ are the nodes of a rational sextic if the four cubics on the last six points P_6^2, and respectively each three of the first four points, meet again by pairs in six points which also lie on a cubic curve containing P_6^2.*

An immediate proof is as follows: Let u be the canonical elliptic parameter on the cubic $C^{(1)}$; $u_{6,2}$, the sum of the parameters of P_6^2 on $C^{(1)}$; u_2, u_3, u_4, the parameters of p_2, p_3, p_4; and u_{12}, u_{13}, u_{14}, the parameters of the intersections with $C^{(1)}$ of $C^{(2)}, C^{(3)}, C^{(4)}$ respectively. Then the intersections of $C^{(1)}$ with $C^{(2)}, C^{(3)}, C^{(4)}$, and $C^{(0)}$ are all given; and

$$u_{6,2} + u_3 + u_4 + u_{12} \equiv 0, \quad u_{6,2} + u_4 + u_2 + u_{13} \equiv 0,$$
$$u_{6,2} + u_2 + u_3 + u_{14} \equiv 0, \quad u_{6,2} + u_{12} + u_{13} + u_{14} \equiv 0.$$

Hence $2(u_{6,2} + u_2 + u_3 + u_4) \equiv 0$, and the nine nodes on $C^{(1)}$ are a half-period set. If three such sets of nine are half-period sets, the set of ten is the nodal set of a rational sextic.

The configuration of five rational sextics $S^{(i)}(\tau)$ is thus projectively equivalent to an inscribed five-plane of a cubic surface Γ^3 together with an isolated line six on Γ^3. We consider further the configuration of ten points r_{ij} in the plane of $\bar{K}(t)$ without reference to the points R_6^2. From the linear relations which connect $x_1(x_2 - x_3)$, $x_2(x_3 - x_1)$, $x_3(x_1 - x_2)$; $(x_2 - x_3)$, $(x_3 - x_1)$, $(x_1 - x_2)$; x_i, x_j, $(x_j - x_i)$ $(i, j = 1, 2, 3)$, respectively we find a symmetrical set of five quadratic relations connecting the binary quadratics r; namely

$$(7) \qquad R^{(m)} \equiv r_{jk}r_{il} + r_{ki}r_{jl} + r_{ij}r_{kl} \equiv 0.$$

If these are satisfied by the ten quadratics, they will serve for the representation (1) of a rational sextic.

We wish to prove that the factors of proportionality in the ten quadratics r_{ij} can be chosen in such wise that the five quadratic relations (7) can be replaced by the five linear relations,

$$(8) \qquad L^{(m)} \equiv r_{mi} + r_{mj} + r_{mk} + r_{ml} \equiv 0,$$

themselves linearly related, and by one of the five quadratic relations R. We may assume that these factors have been chosen so that

$$(9) \qquad L^{(4)} \equiv r_{40} + r_{41} + r_{42} + r_{43} \equiv 0,$$

since any change in these factors can be compensated in (1) by a corresponding change in the factors attached to the remaining r_{ij}. We shall still have at disposal a factor common to these six. Let now

$$(10) \qquad r_{30} + r_{31} + r_{32} \equiv r.$$

From this and $R^{(1)}, R^{(0)}$ by multiplying respectively by $r_{04}r_{42}$, $r_{24} + r_{14}$, r_{04} and adding, the r_{31} and r_{32} are eliminated to yield

$$r_{30}[r_{04}r_{42} + r_{42}(r_{24} + r_{14})] \equiv r\, r_{04}r_{42} + r_{34}(r_{02}r_{24} + r_{02}r_{14} + r_{21}r_{04}).$$

On applying $R^{(3)}$ to bring into evidence the factor r_{24}, this becomes, after using (9),

$$(11) \qquad r_{43}(r_{30}+r_{20}+r_{10}) \equiv r\, r_{04}.$$

Since r_{04} and r_{43} have no common factor, r must be proportional to r_{43}, and the factor still undisposed in r_{30}, r_{31}, r_{32} in (10) may be so chosen that $r = r_{43}$. Then (10) reduces to $L^{(3)} \equiv 0$, and from (11) there follows that $L^{(0)} \equiv 0$. A similar elimination of r_{30}, r_{32}, rather than r_{31}, r_{32}, and the same choice of the factor in r, yields $L^{(1)} \equiv 0$. Then also $L^{(2)} \equiv 0$, since

$$(12) \qquad L^{(0)} + L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)} \equiv 0.$$

Conversely the five linear relations and one quadratic relation yield at once the entire set of five quadratic relations.

The ten points r_{ij}, as a planar set, are determined projectively by the linear relations (8), and then any one of the quadratic relations (7) determines the position of the norm-conic, $\bar{K}(t)$. For, the symmetric form,

$$r_{jk}(t)\cdot r_{il}(t') + r_{jk}(t')\cdot r_{il}(t) + r_{ki}(t)\cdot r_{jl}(t') \\ + r_{ki}(t')\cdot r_{jl}(t) + r_{ij}(t)\cdot r_{kl}(t') + r_{ij}(t')\cdot r_{kl}(t),$$

vanishes when $t = t'$, and therefore is $k\cdot(tt')^2$, i. e., equated to zero, it is the equation of $\bar{K}(t)$. We denote the points r_{ij}, whose coordinates are the coefficients of the quadratics r_{ij}, by $q_1, \cdots, q_9, q_0$, and seek the conditions on the points which imply that such factors of proportionality can be attached to them that the linear relations (8) are satisfied when $r_{ij} = -r_{ji}$. Specifically let respectively

$$r_{01}, r_{02}, r_{03}, r_{04}, r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34} \\ = \lambda_1(q_1\xi), \lambda_2(q_2\xi), \cdots\cdots, \lambda_9(q_9\xi), \lambda_0(q_0\xi).$$

Let (ijk) denote the determinant of the coordinates of the points q_i, q_j, q_k. Then the relations (8) that must be satisfied by the points q are

$$\begin{aligned}
L^{(0)} &\equiv \phantom{(567)(q_1\xi)+{}} (234)(q_1\xi) + (143)(q_2\xi) + (124)(q_3\xi) + (132)(q_4\xi) \equiv 0,\\
L^{(1)} &\equiv (567)(q_1\xi) + (176)(q_5\xi) + (157)(q_6\xi) + (165)(q_7\xi) \equiv 0,\\
L^{(2)} &\equiv (589)(q_2\xi) + (298)(q_5\xi) + (259)(q_8\xi) + (285)(q_9\xi) \equiv 0,\\
L^{(3)} &\equiv (680)(q_3\xi) + (308)(q_6\xi) + (360)(q_8\xi) + (386)(q_0\xi) \equiv 0,\\
L^{(4)} &\equiv (790)(q_4\xi) + (409)(q_7\xi) + (470)(q_9\xi) + (497)(q_0\xi) \equiv 0.
\end{aligned}$$

We now multiply $L^{(0)}$ by (567), and $L^{(1)}$ by (243) to ensure that $r_{10} = -r_{01}$; then $L^{(0)}$ and $L^{(1)}$ by (589), and $L^{(2)}$ by (134)(567) to ensure that $r_{02} = -r_{20}$; then $L^{(0)}, L^{(1)}, L^{(2)}$ by (680), and $L^{(3)}$ by (142)(567)(589) to ensure that $r_{03} = -r_{30}$; and finally $L^{(0)}, L^{(1)}, L^{(2)}, L^{(3)}$ by (790), and $L^{(4)}$ by (123)(567)(589)(680) to ensure that $r_{04} = -r_{40}$. Then the condition $r_{12} = -r_{21}$ yields the relation $(143)(567)(289) = (167)(589)(243)$, which expresses that the triangle whose vertices are q_1, q_2, q_5 is perspective to the triangle whose sides are q_8q_9, q_6q_7, q_3q_4. The other conditions have a

similar translation, as might be inferred from the symmetry; whence, on returning to the original notation [cf. A. G., **56** (b)]:

(13) *The ten points r_{ij} can be separated in* 10 *ways, by isolating a point r_{ij}, into three points r_{lm}, r_{mk}, r_{kl}, and three pairs of points on lines $r_{ik}r_{jk}, r_{il}r_{jl}, r_{im}r_{jm}$ such that the triangles of the three points and the three lines are perspective.*

The perspectivity of these triangles, for isolated r_{34}, may be derived from the quadratic identities $R^{(3)}$, $R^{(4)}$ in (7). We have remarked that each of these furnishes the norm-conic, $\bar{K}(t)$. Let the first yield $k_3\bar{K}$, and the second $k_4\bar{K}$. From the combination $k_4R^{(3)} - k_3R^{(4)}$ we get the following identity connecting three pairs of points:

$$r_{12}(k_3r_{03} + k_4r_{04}) + r_{20}(k_3r_{13} + k_4r_{14}) + r_{01}(k_3r_{23} + k_4r_{24}) \equiv 0,$$

a ternary identity valid throughout the plane, and not merely on $\bar{K}(t)$. This says that the three pairs are the vertices of a four-line, and that the points,

$$k_3r_{03} + k_4r_{04},\ k_3r_{13} + k_4r_{14},\ k_3r_{23} + k_4r_{24},$$

are three points of a line which is the axis of perspectivity. For, the three points, $r_{12}, r_{20}, k_3r_{23} + k_4r_{24}$ also are on a line, i. e., the lines $r_{12}r_{20}$ and $r_{23}r_{24}$ meet on the axis.

Since $r_{12}, r_{20}, k_3r_{23} + k_4r_{24}$ are on a line they are linearly related, and $k_3 = k_4$ due to $L^{(2)} \equiv 0$. Similarly all the constants k are equal, whence:

(14) *If the points r_{ij} have such factors of proportionality that the linear relations* (8) *are satisfied, then the left members of the quadratic relations* (7) *are identical, and each represents the reference conic.*

A linear construction for the configuration r_{ij} is indicated. Let eight of the points, say all except r_{24} and r_{34}, be selected arbitrarily, and consider the pair of perspective triangles determined by isolating r_{34}. The side $r_{01}r_{02}$ meets the side $r_{03}r_{04}$ in a point a; the side $r_{10}r_{12}$ meets the side $r_{13}r_{14}$ in a point b, whence ab is the axis ξ of perspection. The side $r_{21}r_{20}$ meets ξ in a point c, and r_{24} must be a point on the line $r_{23}c$. After selecting r_{24} on this line, a similar construction with the pairs of perspective triangles obtained by isolating first r_{24}, and then r_{23}, determines a pair of lines on r_{34}. Thus the configuration is subject to three independent projective conditions and depends upon nine absolute projective constants.

If the sextic $\bar{S}_2(t)$ is mapped as in A. G., **56** (9) upon a rational octavic in space with nine actual nodes, then five of the pairs of nodal parameters, and the ten pairs of parameters of further points on a plane containing three of the five nodes, yield a set of 15 quadratics which have algebraic properties analogous to those of the 10 r_{ij}'s.

URBANA, ILLINOIS.

A Contribution to the Theory of Capacity.*

BY OLIVER D. KELLOGG AND FLORIN VASILESCO.

1. *Introduction.* The word *domain* will be used in the sense of open continuum. For definiteness, we shall restrict our considerations to space of three dimensions. A domain T is a *Dirichlet domain* if, corresponding to any continuous boundary values, there exists a function, harmonic in T (and this includes the demand that it shall vanish at infinity, if T contains the infinite region), and assuming the given boundary values. If T is infinite, but has a boundary t which is bounded, the *conductor potential* of t is the function V which is harmonic in T and assumes the boundary values 1. The *capacity* of t is the total charge on t producing the potential V, or

$$c = -\frac{1}{4\pi} \int\!\!\int_S \frac{\partial V}{\partial n}\, dS,$$

where S is any smooth surface enclosing t, and n is the outward normal.

The notion of capacity may be defined for any bounded set E.† If E' denotes the set consisting of E and its limit points, the complement of E' is open, and so consists of a denumerable set of domains. One and only one of these will be an infinite domain T. The complete boundary t of T will lie in E'. If $T_1, T_2, T_3, \cdots$ denotes an infinite set of nested Dirichlet domains in T, such that any point of T belongs ultimately to T_n, the conductor potential V_n of the boundary of T_n decreases monotonely, as n becomes infinite, to a function V which is harmonic in T. If a different sequence of Dirichlet domains $T'_1, T'_2, T'_3, \cdots$ is used, the limiting function V' is the same as V. For on the one hand, since $V' \leqq 1$ on the boundaries of $T_1, T_2, T_3, \cdots$, it cannot exceed the corresponding conductor potentials, and hence $V' \leqq V$. On the other hand, $V \leqq V'$, and so $V' = V$. Thus V is characteristic of the domain T, and not of the method by which it is obtained. We call it the *conductor potential* of E, and the corresponding charge, as given by Gauss' integral, the *capacity* of E. We remark that by a theorem of Harnack, the conductor potential V_n and its derivatives of any given order, converge uniformly to V and its corresponding derivatives, in any closed region in T. It

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* Read before the American Mathematical Society, December 27, 1928.

† Wiener, *Journal of Mathematics and Physics, Massachusetts Institute of Technology*, Vol. 3, p. 49, p. 127.

follows that the capacity of t is the limit of the capacities of the boundaries of the domains $T_1, T_2, T_3, \cdots$.

The points of t at which V approaches 1 are called *regular* points of t. All other points of t are called *exceptional.* These points are also characterized as points at which Green's function, similarly defined as the limit of a sequence, does, or does not, approach 0.

Wiener * has given the following criterion for the regular or exceptional character of a boundary point p. Let λ denote a positive proper fraction, let $\lambda^n(p)$ denote the sphere about p of radius λ^n. Let Γ denote the complement of T, and $\Gamma_n(p)$ the points of Γ in the closed region between the two spheres $\lambda^{n+1}(p)$ and $\lambda^n(p)$. Let $\gamma_n(p)$ be the capacity of $\Gamma_n(p)$. *Then p is regular or exceptional according as the series*

$$\sum_0^\infty \frac{\gamma_n(p)}{\lambda^n} \tag{1}$$

diverges or converges.

If T is any domain whose boundary t is bounded, and if $F(p)$ is any function, defined and continuous on t, there exists a function U, bounded and harmonic in T, which takes on the values $F(p)$ at every regular point of t.† If T is a Dirichlet domain, U is uniquely determined. The question naturally arises as to whether this is true for any domain; that is, *can there exist more than one function, bounded and harmonic in T, and taking on the same preassigned continuous boundary values at every regular boundary point?*

This question would be definitely answered in the negative provided the following lemma ‡ were established:

Fundamental Lemma. Any bounded closed set of points of positive capacity contains at least one regular point.

Conversely, the theorem of uniqueness implies the validity of the lemma.§ The lemma is of interest not only because of its connection with the theorem of uniqueness, but because many other properties of harmonic functions with respect to behavior on the boundary depend on it.‖ The corresponding lemma

* *L. c.*, p. 130. See also Kellogg, *Bulletin of the American Mathematical Society*, Vol. 32 (1926), pp. 616-620.

† Kellogg, *Proceedings of the American Academy*, Vol. 58 (1923), pp. 528-529; Wiener, *l. c.*, p. 5.

‡ Kellogg, *Proceedings of the National Academy of Sciences*, Vol. 12 (1926), p. 406.

§ Kellogg, *Acta Universitatis, Szeged*, Vol. 4 (1928), pp. 4, 5. In this note it is regarded as part of the definition of a harmonic function in T that it shall vanish at infinity if T is infinite.

‖ See Vasilesco, *Journal de Mathématiques*, in a number soon to appear.

in two dimensions has been established.* Essential differences arise in three dimensions, and the truth of the lemma here is in doubt. The present paper offers results which reduce the scope of the problem, and which may well have inherent interest.

2. *Transformations of the Criterion of Wiener.* If a set of points is contained in a sphere, we may define the *relative capacity* of the set as the ratio of its capacity to that of the sphere—namely to the radius. Thus the relative capacity depends not only on the set, but on some sphere as well. Usually it will be clear what sphere is meant without explicit statement.

It will be noticed that the nth term of the series (1) is the relative capacity of $\Gamma_n(p)$. By a consideration of the conductor potential, one verifies at once that if a set of points and a containing sphere be subjected to a homothetic transformation, the relative capacity remains unchanged. We thus arrive at a form of Wiener's criterion given by Vasilesco: † let c_n denote the capacity of the set obtained by subjecting the set $\Gamma_n(p)$ to a homothetic transformation with center p and ratio of similitude $\lambda^n : 1$. *Then p is regular or not according as*

$$\sum_0^\infty c_n \tag{2}$$

diverges or converges.

Suppose now that to a given bounded set E there corresponds a point p and a number λ, $0 < \lambda < 1$, such that the set homothetic to the portion of E in the sphere of radius λ about p contains E, the center and ratio of similitude being p and $\lambda : 1$, respectively. It follows that the same is true for the set homothetic to the portion of E in the sphere of radius λ^n about p, the ratio of similitude being $\lambda^n : 1$. Then *either p is regular, or E is of capacity* 0. For $c_0 \leqq c_1 \leqq c_2 \leqq \cdots$ (see the inequalities (4), below), so that the series (2) diverges, unless all its terms are 0. Thus the fundamental lemma is true for sets of this type, which includes such sets as the surfaces of cones and triangles, and sets formed after the analogy of Cantor's nowhere dense perfect set.‡

It is frequently convenient, in the application of Wiener's criterion to use, not the sets $\Gamma_n(p)$, but the sets of points $\Delta_n(p)$ of Γ in the closed spheres $\lambda^n(p)$. Let $\delta_n(p)$ denote the capacity of $\Delta_n(p)$. *Then p is regular or not according as the series*

* Kellogg, *Comptes Rendus*, Vol. 187 (1928), p. 526.

† *Comptes Rendus*, Vol. 186 (1928), p. 23.

‡ See Kellogg, *l. c. Proceedings of the American Academy;* Wiener, *l. c.*, p. 145.

$$(3) \qquad \sum_{0}^{\infty} \frac{\delta_n(p)}{\lambda^n}$$

diverges or converges.

To prove this, we remark that if $c(e)$ denotes the capacity of the set e, then for any two bounded sets e' and e'',

$$(4) \qquad c(e') \leqq c(e'+e'') \leqq c(e') + c(e'').$$

It follows that

$$\gamma_n(p) \leqq \delta_n(p), \text{ and } \delta_n(p) \leqq \gamma_n(p) + \delta_{n+1}(p).$$

From the first inequality it follows that (3) diverges if (1) does. If the second inequality be divided by λ^n and summed from 0 to n, we have, since a relative capacity never exceeds 1,

$$\sum_{0}^{n} \frac{\gamma_n(p)}{\lambda^n} \geqq (1-\lambda) \sum_{0}^{n} \frac{\delta_n(p)}{\lambda^n} - \lambda.$$

Accordingly, the series (3) converges if (1) does. This proves the theorem.

An advantage of this transformation of the criterion is that it throws into evidence the following fact: we may replace the spheres $\lambda^n(p)$ by regions $\sigma_n(p)$ of quite varied character. Thus if $\sigma_n(p)$ has the property that a positive integer m exists, such that from some n on $\sigma_n(p)$ is contained in $\lambda^{n-m}(p)$ and contains $\lambda^{n+m}(p)$, we may interpret $\delta_n(p)$ as the capacity of the portion of Γ in the closed region $\sigma_n(p)$ in the above theorem. Because of their space-filling property, it may be convenient to use cubes.

If λ is fixed, and if a bounded set E is given, determining a closed bounded set Γ, the series

$$(3) \qquad W(P) = \sum_{0}^{\infty} \frac{\delta_n(P)}{\lambda^n},$$

where $\delta_n(P)$ means the capacity of $\Delta_n(P)$, defines a function $W(P)$ at every point of space, if we admit $W(P) = \infty$ as a possible definition, characterizing a point at which the series diverges. With respect to this function, we shall prove a number of theorems, the first of which is as follows:

THEOREM I. *If $W(P_0) = \infty$, then given N, however large, there is a neighborhood of P_0 at every point of which $W(P) > N$.*

To show this, we select n so that

$$[\delta_0(P_0)/1] + [\delta_1(P_0)/\lambda] + \cdots + [\delta_n(P_0)/\lambda^n] > (N/\lambda) + 1.$$

Let $\rho = \lambda^n - \lambda^{n+1}$. Then if P is confined to the sphere of radius ρ about P_0, the points of Γ in the sphere $\lambda^n(P)$ include all those in the sphere $\lambda^{n+1}(P_0)$.

Hence $\delta_n(P) \geqq \delta_{n+1}(P_0)$, and similarly $\delta_{n-1}(P) \geqq \delta_n(P_0), \cdots, \delta_0(P) \geqq \delta_1(P_0)$. Hence

$$\begin{aligned} W(P) &\geqq [\delta_0(P)/1] + [\delta_1(P)/\lambda] + \cdots + [\delta_n(P)/\lambda^n] \\ &\geqq [\delta_1(P_0)/1] + [\delta_2(P_0)/\lambda] + \cdots + [\delta_{n+1}(P_0)/\lambda^n] \\ &\geqq \lambda[(N/\lambda) + 1 - \delta_0(P_0)] > N, \end{aligned}$$

as was to be proved. From this follows the

Corollary. If $W(P)$ is bounded on a set of points everywhere dense on E, E can have no regular points.

We close this section with an integral form for the criterion for regularity. Let $c(\rho)$ denote the capacity of the portion of Γ in the closed sphere of radius ρ about p. Then $c(\rho)$ is a monotone increasing bounded function of ρ, and so is integrable. Moreover,

$$\int_{\lambda^{n+1}}^{\lambda^n} \frac{c(\rho)}{\rho^2}\, d\rho \leqq c(\lambda^n) \int_{\lambda^{n+1}}^{\lambda^n} \frac{d\rho}{\rho^2} = \delta_n(p) \left(\frac{1}{\lambda^{n+1}} - \frac{1}{\lambda^n}\right) \leqq \frac{1}{\lambda} \cdot \frac{\delta_n(p)}{\lambda^n}\ .$$

On the other hand,

$$\int_{\lambda^{n+1}}^{\lambda^n} \frac{c(\rho)}{\rho^2}\, d\rho \geqq c(\lambda^{n+1}) \left(\frac{1}{\lambda^{n+1}} - \frac{1}{\lambda_n}\right) = (1-\lambda)\ \frac{\delta_{n+1}(p)}{\lambda^{n+1}}\ .$$

Hence, summing the inequalities, we obtain

$$(1-\lambda)\ [W(p) - \delta_0(p)] \leqq \int_0^1 \frac{c(\rho)}{\rho^2}\, d\rho \leqq (1/\lambda) W(p).$$

Thus p is regular or not according as the integral (in general improper)

$$\int_0^1 \frac{c(\rho)}{\rho^2}\, d\rho$$

is divergent or not.

The integral, like the series, is a function defined for any point P of space. With respect to either function, the fundamental lemma may be expressed: *given a closed bounded set E, the function is either infinite at some point of E, or else it is identically* 0. It may be remarked that if E either has interior points, or bounds any domains other than the infinite one—in other words, if Γ has interior points—the lemma holds for E. For it is obvious that a point p of the boundary of Γ, on a sphere in Γ, is a regular point.

3. *Capacity as a Function of Sets of Points.* For the sequel some further concepts will be useful.* A set E will be said to be *of capacity* 0 *at a*

* Vasilesco, *Comptes Rendus*, Vol. 185 (1927), p. 1572; *Journal de Mathématiques, l. c.*

point P if there is a sphere about P the portion of E within which has the capacity 0. A closed set is said to be *reduced* provided it is non-empty and is of capacity 0 at none of its points. It has been shown * that every closed bounded set E of positive capacity contains a closed reduced set, and that it may be obtained by removing from E the points at which E is of capacity 0. This process does not change the capacity or the conductor potential of the set. Thus, for the purposes of establishing or disproving the fundamental lemma, one may restrict oneself to closed reduced sets.

The following measure of closeness of two sets is well adapted to the study of capacity.† We say that two closed bounded sets e and e' *differ by at most* ϵ, if e' is contained in the set I_ϵ consisting of the points within and on the spheres of radius ϵ, one about each point of e, and if e is contained in the corresponding set I_ϵ' formed for e'.

With these preliminaries, we establish the following theorem:

THEOREM II. *Let E denote a closed bounded set, and e any closed part of E. Then the capacity $c(e)$ of e is a function of sets which is upper semicontinuous. It is continuous only if* $c(E) = 0$.

To prove this, let any positive number η be given. We form the set of spheres $I_{1/n}$ of radius $1/n$ about the points of e. As we saw in connection with the definition of capacity in § 1, $c(I_{1/n})$ approaches the limit $c(e)$ as n becomes infinite. There is therefore an integer n such that

$$c(I_{1/n}) < c(e) + \eta.$$

If e' is any set in $I_{1/n}$, and, in particular, if e' is any set differing from e by less than $1/n$, $c(e') \leqq c(I_{1/n})$, by the inequalities (4), and hence

$$c(e') < c(e) + \eta,$$

so that $c(e)$ is upper-semicontinuous, as stated.

To establish the second part of the theorem, we notice that for any n there is always a set e' of capacity 0 differing from e by less than $1/n$. To see this, we note that by the Heine-Borel theorem, the points of e are interior to a finite number of the spheres of $I_{1/2n}$. If we choose for e' a single point of E in each of these spheres, e and e' will differ by at most $1/n$, and e' will have the capacity 0. Thus there are sets of capacity 0 arbitrarily near to e,

* Vasilesco, *Journal de Mathématiques, l. c.*

† Vasilesco, *Essai sur les fonctions multiformes de variables réelles*, Thèse, Paris, Gauthier-Villars (1925), p. 7.

and $c(e)$ can only be continuous if $c(e) = 0$ for every closed part of E, and in particular, for E itself. Thus the theorem is established.

4. *The Function Defined by the Wiener Series.* We begin by considering a typical term of the Wiener series, formed for a *reduced bounded set* E, and establish

THEOREM III. *The function $\delta_n(P)$, defined at every point of space, is upper semi-continuous, and in consequence, either continuous or of the first class, in the sense of Baire. It is not, in general, continuous.* The same holds for $\gamma_n(P)$.

For if P and η are given, and the set e of Theorem II is identified with $\Delta_n(P)$, then $\delta_n(P) = c(e)$, and so for all e' in $I_{1/m}$, for sufficiently large m,

$$c(e') < \delta_n(P) + \eta.$$

The set of points of E at a distance $1/m$ or more from the points of $\Delta_n(P)$ is a closed set with no points in $\lambda^n(P)$. If ρ' denotes its distance from $\lambda^n(P)$, and if ρ is the less of the two numbers $1/m$ and ρ', then for $P'P < \rho$, the set $\Delta_n(P')$ lies in $I_{1/m}$. Hence

$$\delta_n(P') < \delta_n(P) + \eta \text{ for } P'P < \rho,$$

and $\delta_n(P)$ is upper semi-continuous.

An example in which $\delta_n(P)$ is discontinuous is that in which E is the smaller portion of a sphere of radius λ^n bounded by a small circle. If P is the center of the sphere, $\delta_n(P)$ is positive, whereas for all points P' on the axis of the small circle and farther from the circle than P, $\delta_n(P') = 0$. Such a situation may arise in any case in which E, or the associated set Γ (consisting of E and any finite domains it may bound), has sub-sets of positive capacity on a sphere $\lambda^n(P)$ which are not limit points of points of Γ in the interior of the sphere. Thus if we increase the set E of the previous example by adjoining a small cone or cylinder along the diameter of the sphere which is the axis of the circle, thereby forming a new reduced set E, the function $\delta_n(P')$ will still fail to be continuous, even while P' is a point of E itself.

We shall revert to this question presently. First let us point out a corollary of the last theorem.

Corollary. The series of Wiener—(1) *or* (3)*—is a function of P of class 2 or lower, in the sense of Baire.*

For, since $\delta_n(P)$ is of class 1 or lower, the same is true of the separate

terms of the series, and thus of the partial sums. Hence $W(P)$, the limit of the sum of the first n terms, is of class 2 or lower.

Nevertheless, there is a considerable category of sets E for which $W(P)$ is of class not greater than 1. This is shown in the following theorem.

THEOREM IV. *If E is such that the set of points common to the associated set Γ and the surface of any sphere $\lambda^n(P)$, P a point of t, is of capacity 0, then E, Γ, and t are identical, $\delta_n(P)$—and similarly $\gamma_n(P)$—is continuous, and $W(P)$ is at most of class one on E.*

Suppose that $\Gamma \not\equiv t$; let M be a point of Γ not belonging to t. As $t+T$ is a closed set of points, there exists a sphere of small radius about M whose points do not belong to $t+T$; they belong to Γ. Consider then a sphere (λ^n) of smaller radius λ^n (n large enough) about M and the straight line MM' joining M to a point M' of T. Since the sphere (λ^n) lies in Γ, the same is the case for all the spheres of the same radius about every point M'' near to M. As M'' moves away towards M' there will be a first point M_1 on MM' such that the corresponding sphere will have a point P of t on its surface. Then the sphere $\lambda^n(P)$ would contain on its surface a set of points of Γ of positive capacity. But this is contrary to hypothesis.

Accordingly $\Gamma \equiv t$, and *a fortiori* $E \equiv t$, for $\Gamma > E > t$.

To prove the remainder of the theorem, let P be any point of E, and $\lambda^n(P)$ any of the spheres of the theorem with P as center. Let $s_n(P)$ denote the set of points of E on the surface of $\lambda^n(P)$. As $s_n(P)$ has the capacity 0, by hypothesis, there corresponds to any $\eta > 0$ a set I_δ of spheres of radius δ about the points of $s_n(P)$, whose capacity is less than η. The points of E in $\lambda^n(P)$, but not interior to I_δ, form a closed set e^* with no points on the surface of $\lambda^n(P)$. Let ρ be the distance from e^* to the surface of $\lambda^n(P)$. Then if $P'P < \rho$, $\lambda^n(P')$ will contain all points of e^*, and hence

$$\delta_n(P') \geqq c(e^*).$$

Also, $e^* + I_\delta$ contains $\Delta_n(P)$, and hence

$$c(e^*) + \eta \geqq \delta_n(P).$$

It follows from the two inequalities that

$$\delta_n(P') \geqq \delta_n(P) - \eta,$$

and $\delta_n(P)$ is thus lower semi-continuous at P. Accordingly, by Theorem III, it is continuous at P. As P was any point, $\delta_n(P)$ is everywhere continuous. We remark for later use, that if, for a certain n and P, $s_n(P)$ has the

capacity zero, the proof shows that $\delta_n(P)$ is continuous at the center of that sphere.

Sets of the type contemplated in the above theorem are easy to form. Among them are plane sets, sets on spheres, and sets on any other analytic surfaces. For the intersections of such sets with a sphere $\lambda^n(P)$ would lie on analytic curves, and these have the capacity 0. It is also possible that $\delta_n(P)$ is discontinuous and $W(P)$ of class 2 for some value of λ, whereas for a neighboring value $\delta_n(P)$ is continuous for all n, and $W(P)$ of class not greater than 1. A general method of constructing sets for which the $\delta_n(P)$ are continuous is given by the

Corollary. If for the closed bounded set E, the functions $\delta_n(P)$, P being a point of t, are not all continuous, then E contains a reduced subset E such that the functions $\delta_n(P)$, P being a point of E', are continuous and $W(P)$ is of class not greater than 1 *on E'.*

Suppose first that $\Gamma \equiv t$ and therefore $\Gamma \equiv E$; and suppose that for some fixed n, $\delta_n(P)$ is discontinuous at P_0. The part of E on the surface of $\lambda^n(P_0)$ is then of positive capacity. If this set is reduced, the resulting set E' is not of zero capacity and has the desired property, for it lies on a sphere of radius λ^n.

If now $\Gamma \not\equiv t$, the set Γ contains points which are not limiting points of T and are therefore interior points of Γ. Hence Γ contains at least one domain whose boundary e is a part of t. If P is a point of e and M a point of the domain just mentioned, the portion of e determined by the sphere of center P and radius PM is a set of positive capacity.* This portion e_1 of e is itself the complete boundary of a certain infinite domain T_1, such that e_1 coincides with its corresponding Γ. The proof of the first part of the theorem accordingly applies to e_1, which contains therefore a subset E' of the kind specified.

Since the function $\delta_n(P)$ is continuous on E', the function $W(P)$ will be of class not greater than 1 on E'.

If now the functions $\delta_n(P)$ are all continuous, $W(P)$ is not merely of class 1 or lower, but more, it is lower semi-continuous. This is a consequence of the fact that the terms of the Wiener series are never negative. It follows that if N is any positive number, the points where $W(P) \leqq N$ form a closed set. It is not true that the points where $W(P)$ is infinite always form a closed set, for the Lebesgue spine † gives a closed set for

* Vasilesco, *Journal de Mathématiques, loc. cit.* No. 27.

† "Sur des cas d'impossibilité du problème de Dirichlet," *Comptes Rendus de la Société Mathématique de France* (1923), p. 17.

which $W(P)$ is infinite at all but a single point, namely the point of the spine. The fundamental lemma has to do with the possibility of bounded closed sets at which $W(P)$ is nowhere infinite and yet not identically 0. If such a set exists, it must coincide with the associated sets Γ and t. For the existence of an interior point of Γ would imply a regular point of E, and at it $W(P)$ would be infinite.*

If a closed bounded set E without regular points exists, we may, by the corollary, assume that the sub-set, on which the corresponding $W(P)$ does not exceed any assigned positive number, is closed. If then E_n denote the sub-set on which $W(P) \leqq n$, we have, since $W(P)$ is nowhere infinite,

$$E = E_1 + E_2 + E_3 + \cdots.$$

It follows that for some n, $c(E_n) > 0$.† As $W(P)$, formed for E, is not greater than n at the points of E_n, it is certainly not greater than n at the points of the set E', obtained from E_n by reducing it according to the process of the last section. It follows that $W(P)$, formed for the sub-set E' of E, must also be not greater than n at the points of E'.

We shall now show that this function $W(P)$ is bounded. As it is bounded by n on E', it remains to show that it is bounded elsewhere. Let P be any point not in E', and ρ its distance from the nearest point p of E'. Let $\lambda^{k+1} < \rho \leqq \lambda^k$. We may assume k positive or 0, for otherwise $\Delta_0(P)$ would be empty, and $W(P) = 0$. Now $\Delta_{k+1}(P)$, $\Delta_{k+2}(P), \cdots$ are empty, so that $\delta_{k+1}(P) = \delta_{k+2}(P) = \cdots = 0$.

Moreover, for $i \leqq k$, $\Delta_i(P)$ is contained in the sphere about p of radius $\lambda^i + \lambda^k = \lambda^i(1 + \lambda^{k-i}) \leqq 2\lambda^i \leqq \lambda^{i-s}$, if s is the first integer for which $2 \leqq \lambda^{-s}$. That is, $\Delta_i(P)$ is a part of $\Delta_{i-s}(p)$. Hence if $s \leqq i \leqq k$, $\delta_i(P) \leqq \delta_{i-s}(p)$, and as $\delta_i(P) = 0$ for $i > k$, this inequality holds for all $i \geqq s$. We have, therefore, since relative capacities never exceed 1,

$$\begin{aligned} W(P) &= \frac{\delta_0(P)}{1} + \cdots + \frac{\delta_{s-1}(P)}{\lambda^{s-1}} + \frac{\delta_s(P)}{\lambda^s} + \frac{\delta_{s+1}(P)}{\lambda^{s+1}} + \cdots \\ &\leqq 1 + \cdots + 1 + 1/\lambda^s \left[\frac{\delta_0(p)}{1} + \frac{\delta_1(p)}{\lambda} + \cdots \right] \\ &\leqq s + n/\lambda^s, \end{aligned}$$

and so $W(P)$ is bounded.

Two functions $W(P)$, formed for the same set of points, but with

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* Vasilesco, *Journal de Mathématiques, loc. cit.*, No. 38.

† If the sum of a denumerable set of closed sets of capacity 0 is closed, it has the capacity 0. Vasilesco, *Journal de Mathématiques, l. c.*, § 8.

different values of λ, are bounded each by a linear function of the other, so that if either is bounded, the other is also. Combining the results here attained with the last corollary, we have

THEOREM V. *If there exists a bounded closed set E, of positive capacity, without regular points, then there exists a reduced sub-set E' for which the series of Wiener is bounded, lower semi-continuous, and all of whose terms are everywhere continuous.*

By means of a theorem of Baire, the last theorem admits an alternative proof. If the set of the theorem exists, there is, as we have seen, a reduced set also satisfying the hypothesis, which, by the last corollary, we may assume to be such that $W(P)$ is of class not greater than 1. Accordingly, if we denote this set by E, the points of continuity of $W(P)$, by the theorem of Baire, are everywhere dense on E. Let P_0 be a point of continuity of $W(P)$ in E. There is then a sphere σ about P_0 such that when P is in σ, $W(P) < W(P_0) + 1$. Thus $W(P)$ is bounded on the portion of E in σ, and this set has positive capacity, since E is reduced. If now this portion of E be reduced, the resulting set E' has the properties enunciated in the theorem.

5. *The Function Defined by the Integral in the criterion for Regularity.* The integral used in the criterion established in § 2 is

$$J(P) = \int_0^1 \frac{c(\rho, P)}{\rho^2}\, d\rho.$$

We consider, by the side of $J(P)$, the integral

$$J(\alpha, P) = \int_\alpha^1 \frac{c(\rho, P)}{\rho^2}\, d\rho,$$

where α is a fixed number, $0 < \alpha < 1$, and show that when based on a closed bounded set E, this function of P is continuous.

In fact, if $P'P < \delta < \alpha$,

$$c(\rho - \delta, P) \leqq c(\rho, P') \leqq c(\rho + \delta, P),$$

so that

$$\int_\alpha^1 \frac{c(\rho - \delta, P)}{\rho^2}\, d\rho \leqq \int_\alpha^1 \frac{c(\rho, P')}{\rho^2}\, d\rho \leqq \int_\alpha^1 \frac{c(\rho + \delta, P)}{\rho^2}\, d\rho.$$

Now

$$\int_\alpha^1 \frac{c(\rho + \delta, P)}{\rho^2}\, d\rho = \int_{\alpha+\delta}^{1+\delta} \frac{c(\rho, P)}{(\rho - \delta)^2}\, d\rho,$$

and as $c(\rho, P)$ is bounded, this is a continuous function of δ for $-\alpha < \delta < \alpha$. Hence $\delta > 0$ may be chosen so small that

$$| J(\alpha, P') - J(\alpha, P) |$$

is less than any given positive ϵ.

As $J(P) = \lim_{n \to \infty} J(1/n, P)$, it follows that $J(P)$ is lower semi-continuous. We have thus

THEOREM VI. *Given any bounded closed set E, the corresponding function $J(\alpha, P)$ is everywhere continuous, and the function $J(P)$ is lower semi-continuous.*

If the capacity of E is positive, and yet E has no regular points, we may, either by modifying appropriately the proof of Theorem V, or by applying that theorem and noticing that $J(P)$ is bounded by a linear function of $W(P)$, arrive at the following result:

THEOREM V′. *If there exists a closed bounded set E, of positive capacity, without regular points, there exists a reduced sub-set of E for which $J(P)$ is bounded.*

Collineations and Motions in Generalized Spaces.

By M. S. Knebelman.

Introduction. The first four paragraphs of this paper are devoted to the development of the tensor analysis of a space of paths whose connection—affine or projective—is a function of position and direction. The next five paragraphs deal with groups of collineations in a space of paths while the last three deal with groups of motions in a generalized metric space, that is, one in which the components of the fundamental tensor are functions of direction as well as of position.

The theory as developed below deals entirely with finite continuous groups, the fundamental result being the set of theorems which state the necessary and sufficient conditions that a space must satisfy in order to admit an r-parameter group of collineations or motions. Some theorems and properties of groups of motions in a Riemannian space have been generalized so as to apply to the spaces under discussion.

No attempt has been made here to solve the outstanding problem of classifying spaces according to the number and type of collineations that they may admit. The problem of determining spaces admitting an r-parameter translatory group is thus far unsolved as well as the problem of complete integrability of the equations of Killing for a generalized metric space (except for the case of two dimensions).

1. *General space of paths. The affine curvature tensor.* We consider an n-dimensional manifold whose points are represented in a given coordinate system x by the ordered set of numbers $(x^1, \cdots, x^n)$. We have given a system of differential equations

$$(d^2x^i/dt^2) + H^i[x, (dx/dt)] = 0 \qquad (i = 1, \cdots, n), \tag{1.1}$$

where $H^i(x, dx/dt)$ are a set of functions of $x^1, \cdots, x^n$; $dx^1/dt, \cdots, dx^n/dt$ which are positively homogeneous of the second degree in dx^i/dt.

The curves $x^i = f^i(t)$ which satisfy the equations (1.1) are the *paths* of the manifold, the manifold together with the paths constituting a *general space of paths* defined by the functions $H^i(x, dx/dt)$.

The designation "general" is used to distinguish the space of paths as defined above from the case where $H^i(x, dx/dt)$ are polynomials of the second degree in dx/dt and also for the reason that equations (1.1) are the most general system of equations whose integral curves have the property that through any two points in general position, in a properly restricted neighborhood, there passes one and only one path.*

We shall denote dx^i/dt by $\dot{x}^i$ and assume that $H^i(x, \dot{x})$ are analytic in both sets of variables. By Euler's theorem on homogeneous functions we may write equations (1.1) in the form

$$(1.2) \qquad (d\dot{x}^i/dt) + \Gamma^i{}_{jk}(x, \dot{x})\dot{x}^j\dot{x}^k = 0,$$

the summation convention for repeated indices being used and $\Gamma^i{}_{jk}$ being defined by

$$(1.3) \qquad \Gamma^i{}_{jk}(x, \dot{x}) = \tfrac{1}{2}[\partial^2 H^i(x, \dot{x})/\partial\dot{x}^j\,\partial\dot{x}^k].$$

From (1.3) and the definition of $H^i(x, \dot{x})$ it follows that $\Gamma^i{}_{jk}(x, \dot{x})$ are positively homogeneous of degree zero in $\dot{x}$, are symmetric in the subscripts and satisfy the equations

$$(1.4) \qquad (\partial\Gamma^i{}_{jk}/\partial\dot{x}^l) = (\partial\Gamma^i{}_{jl}/\partial\dot{x}^k).$$

The invariant whose components in the given coordinate system x are $\Gamma^i{}_{jk}(x, \dot{x})$ is the *affine connection* of the space of paths, the functions $\Gamma^i{}_{jk}$ being the *coefficients* of affine connection in the given coordinate system. The law of transformation of the Γ's under an arbitrary analytic transformation of coordinates can be obtained from the invariance of form of (1.2) when the affine parameter t is unaltered.† For let $x^i = f^i(\bar{x})$ define a transformation of coordinates; then

$$(1.5) \qquad \dot{x}^i = \frac{\partial x^i}{\partial \bar{x}^\alpha}\dot{\bar{x}}^\alpha, \qquad \frac{d\dot{x}^i}{dt} = \frac{\partial^2 x^i}{\partial\bar{x}^\alpha\,\partial\bar{x}^\beta}\,\dot{\bar{x}}^\beta\dot{\bar{x}}^\alpha + \frac{\partial x^i}{\partial\bar{x}^\alpha}\,\frac{d\dot{\bar{x}}^\alpha}{dt}.$$

Substitution of (1.5) into (1.2) gives, since $(d\dot{\bar{x}}^\alpha/dt) + \bar{\Gamma}^\alpha{}_{\beta\gamma}(\bar{x}, \dot{\bar{x}})\,\dot{\bar{x}}^\beta\dot{\bar{x}}^\gamma = 0$,

$$(1.6) \qquad \left(\Gamma^i{}_{jk}\,\frac{\partial x^j}{\partial\bar{x}^\beta}\,\frac{\partial x^k}{\partial\bar{x}^\gamma} + \frac{\partial^2 x^i}{\partial\bar{x}^\beta\,\partial\bar{x}^\gamma} - \bar{\Gamma}^\alpha{}_{\beta\gamma}\,\frac{\partial x^i}{\partial\bar{x}^\alpha}\right)\dot{\bar{x}}^\beta\dot{\bar{x}}^\gamma = 0,$$

$\Gamma^\alpha{}_{\beta\gamma}$ being the coefficients of affine connection in the coordinate system $\bar{x}$.

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* Cf. O. Veblen, "Remarks on the Foundations of Geometry," *Bulletin of the American Mathematical Society*, Vol. 31 (1925), pp. 121-141; J. Douglass, *Annals of Mathematics*, Vol. 29 (1928), pp. 154-166.

† Cf. J. Douglas, *loc. cit.* for definition of "parameterized paths" and affine parameters.

To simplify (1.6), as well as for future use, we establish the following lemma: *Given a set of analytic functions* $f^{(\alpha)}{}_{(\beta)i_1i_2\dots i_r}(y)$ *which are homogeneous of degree zero in the n independent variables* $y^1, \cdots, y^n$, *symmetric in the indices* $i_1, i_2, \cdots, i_r$—(α) *and* (β) *being any sets of indices—and satisfying the relations*

$$(1.7)\qquad f^{(\alpha)}{}_{(\beta)i_1\dots i_r}\, y^{i_1}\cdots y^{i_r} = 0;\quad \partial f^{(\alpha)}{}_{(\beta)i_1i_2\dots i_r}/\partial y^p = \partial f^{(\alpha)}{}_{(\beta)pi_2\dots i_r}/\partial y^{i_1},$$

then

$$f^{(\alpha)}{}_{(\beta)i_1i_2\dots i_r}(y) = 0.$$

For if we differentiate the first identity of the lemma partially with respect to y^{j_r} we obtain

$$(1.8)\qquad (\partial f^{(\alpha)}{}_{(\beta)i_1\dots i_r}/\partial y^{j_r})\, y^{i_1}\cdots y^{i_r} + r f^{(\alpha)}{}_{(\beta)i_1\dots i_{r-1}}\, y^{i_1}\cdots y^{i_{r-1}} = 0$$

because of the symmetry of f in $i_1 \cdots i_r$. The first term of (1.8) vanishes because of (1.7) and Euler's theorem, hence if we differentiate partially with respect to $y^{j_{r-1}}, y^{j_{r-2}} \cdots y^{j_1}$ we shall obtain $r!f^{(\alpha)}{}_{(\beta)j_1\dots j_r} = 0$, which establishes the lemma.

The quantities in parenthesis of (1.6) are of the form $f^i{}_{\beta\gamma}(\bar{x}, \dot{\bar{x}})$; they obviously satisfy the first two conditions of the lemma, $\dot{\bar{x}}$ being the variables y. To show that they satisfy (1.7) we have from (1.5)

$$(1.9)\qquad \partial x^l/\partial \dot{\bar{x}}^\delta = 0, \qquad \partial \dot{x}^l/\partial \dot{\bar{x}}^\beta = \partial x^l/\partial \bar{x}^\beta, \qquad \partial \dot{x}^i/\partial \bar{x}^\beta = (\partial^2 x^i/\partial \bar{x}^\beta \partial \bar{x}^\gamma)\, \dot{\bar{x}}^\gamma$$

so that

$$(1.10)\qquad \frac{\partial f^i{}_{\beta\gamma}}{\partial \dot{\bar{x}}^\delta} = \frac{\partial \Gamma^i{}_{jk}}{\partial \dot{x}^l}\, \frac{\partial k^j}{\partial \bar{x}^\beta}\, \frac{\partial x^k}{\partial \bar{x}^\gamma}\, \frac{\partial \dot{x}^l}{\partial \dot{\bar{x}}^\delta} - \frac{\partial \bar{\Gamma}_{\beta\gamma}{}^\alpha}{\partial \dot{\bar{x}}^\delta}\, \frac{\partial x^i}{\partial \bar{x}^\alpha} = \frac{\partial f^i{}_{\beta\delta}}{\partial \dot{\bar{x}}^\gamma}$$

in virtue of (1.4). Therefore $f^i{}_{\beta\gamma}$ satisfy all conditions of the lemma and the law of transformation of the coefficients of affine connection may be written as

$$(1.11)\qquad \bar{\Gamma}^\alpha{}_{\beta\gamma}\, \frac{\partial x^i}{\partial \bar{x}^\alpha} = \Gamma^i{}_{jk}\, \frac{\partial x^j}{\partial \bar{x}^\beta}\, \frac{\partial x^k}{\partial \bar{x}^\gamma} + \frac{\partial^2 x^i}{\partial \bar{x}^\beta\, \partial \bar{x}^\gamma}.$$

Since $f^i{}_{\beta\gamma} = 0$ it follows from (1.10) that

$$(1.12)\qquad \bar{\Gamma}^\alpha{}_{\beta\gamma\delta} = \Gamma^i{}_{jkl}\, \frac{\partial x^j}{\partial \bar{x}^\beta}\, \frac{\partial x^k}{\partial \bar{x}^\gamma}\, \frac{\partial x^l}{\partial \bar{x}^\delta}\, \frac{\partial \bar{x}^\alpha}{\partial x^i},$$

where
$$\Gamma^i{}_{jkl} \equiv \partial \Gamma^i{}_{jk}/\partial \dot{x}^l.$$

Since $\Gamma^i{}_{jk}(x, \dot{x})$ are homogeneous of degree zero in $\dot{x}$ we have

$$\Gamma^i{}_{jk}(x, \dot{x}) = \Gamma^i{}_{jk}(x, dx)$$

and it is more convenient in the problems where the affine connection is fundamental to regard the Γ's as functions of x and dx, $dx^1, \cdots, dx^n$ being a set of independent variables subject to the conditions * $d\bar{x}^i = (\partial\bar{x}^i/\partial x^j)dx^j$.

By a *tensor* we shall understand an invariant whose components in a given coordinate system x are functions of x and dx satisfying the tensor law of transformation; that is, we remove the restriction that the components of a tensor must be functions of the coordinates alone.† A tensor whose components depend on x alone will be said to be a *tensor of position* while one depending on both x and dx will be called a *tensor of position and direction.*‡

It thus appears from (1.12) that Γ^i_{jkl} is a tensor. Its vanishing is a necessary and sufficient condition that the affine connection be a function of position. Evidently Γ^i_{jkl} is homogeneous of degree -1 in dx and is symmetric in its subscripts.

By differentiating (1.11) partially with respect to $\bar{x}^\delta$ and eliminating third partial derivatives we get the equations

$$\bar{K}^\alpha_{\beta\gamma\delta} = K^i_{jkl}\frac{\partial x^j}{\partial\bar{x}^\beta}\frac{\partial x^k}{\partial\bar{x}^\gamma}\frac{\partial x^l}{\partial\bar{x}^\delta}\frac{\partial\bar{x}^\alpha}{\partial x^i},$$

where

$$(1.13)\qquad K^i_{jkl} = (\partial\Gamma^i_{jk}/\partial x^l) - (\partial\Gamma^i_{jl}/\partial x^k) + \Gamma^h_{jk}\Gamma^i_{hl} - \Gamma^h_{jl}\Gamma^i_{hk} - (\Gamma^i_{jkh}\Gamma^h_{lm} - \Gamma^i_{jlh}\Gamma^h_{km})dx^m.$$

K^i_{jkl} are the components of the *affine curvature tensor* § of the affine connection. It is evident that K^i_{jkl} are homogeneous of degree zero in dx.

2. *Covariant Differentiation. Fundamental Identities.* By means of the affine connection and its law of transformation it is possible to obtain from any tensor an infinite sequence of tensors. Thus if $\xi^i(x, dx)$ are the components of a vector in the coordinate system x, the components in the coordinate system $\bar{x}$ are given by $\bar{\xi}^i = \xi^\alpha\,\partial\bar{x}^i/\partial x^\alpha$ or

* For a general definition of differentials cf. O. Veblen, *Cambridge Tract*, No. 24, sec. 5.

† Cf. O. Veblen, *loc. cit.*, sec. 8.

‡ It is to be noted that being a tensor of position is a "tensor property" since a transformation of coordinate does not depend on dx. But being a tensor of direction is not a "tensor property," that is, the components of a tensor may be functions of direction in a particular coordinate system without being functions of direction in every coordinate system. There are no tensors whose components are functions of direction alone in every coordinate system.

§ L. Berwald, *Mathematische Zeitschrift*, Vol. 25 (1926), pp. 40-74. Cf. sec. 5, where K^i_{jkl} are obtained from the notion of parallel propagation of a vector around an infinitesimal circuit.

$$\xi^i = \bar{\xi}^\alpha \partial x^i / \partial \bar{x}^\alpha. \tag{2.1}$$

Henceforth we shall use the notation in which a subscript preceded by a period indicates partial differentiation with respect to dx. From (2.1) we have by means of (1.9)

$$\xi^i{}_{.j} = \bar{\xi}^\alpha{}_{.\beta} \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\beta}{\partial x^j} \tag{2.2}$$

from which it follows that $\xi^i{}_{.j}$ are the components of a tensor. From (2.1) we also have

$$\frac{\partial \xi^i}{\partial x^j} = \frac{\partial^2 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} \frac{\partial \bar{x}^\beta}{\partial x^j} \bar{\xi}^\alpha + \frac{\partial x^i}{\partial \bar{x}^\alpha} \left(\frac{\partial \bar{\xi}}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^\beta}{\partial x^j} + \bar{\xi}^\alpha{}_{.\beta} \frac{\partial (d\bar{x}^\beta)}{\partial x^j} \right).$$

Making use of (1.11), (1.9) and (2.2) the last set of equations may be written as

$$\xi^i{}_{,j} = \bar{\xi}^\alpha{}_{,\beta} \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\beta}{\partial x^j}, \tag{2.3}$$

where

$$\xi^i{}_{,j} \equiv \frac{\partial \xi^i}{\partial x^j} + \xi^h \Gamma^i{}_{hj} - \xi^i{}_{.h} \Gamma^h{}_{jk} dx^k. \tag{2.4}$$

From (2.3) it is evident that $\xi^i{}_{,j}$ is a tensor which we call the *covariant derivative* of the vector ξ^i with respect to the affine connection $\Gamma^i{}_{jk}$. And in general if $T^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ are the components of a tensor, its covariant derivative * is given by

$$\begin{aligned} T^{(i)}{}_{(j),k} = {} & \frac{\partial T^{(i)}{}_{(j)}}{\partial x^k} - T^{(i)}{}_{(j).h} \Gamma^h{}_{mk} dx^m \\ & + \sum_{a=1}^{r} T_{(j)}{}^{\cdots i_{a-1} h i_{a+1} \cdots} \Gamma^{i_a}_{hk} - \sum_{a=1}^{s} T^{(i)}{}_{\cdots j_{a-1} h j_{a+1} \cdots} \Gamma^h{}_{j_a k}. \end{aligned} \tag{2.5}$$

In order to obtain the fundamental relations between the components of the curvature tensor we introduce geodesic coordinates defined for an arbitrary point x_0 as origin and arbitrary direction dx_0 by means of

$$x^i = x_0{}^i + \bar{x}^i - \tfrac{1}{2} \Gamma^i{}_{jk}(x_0, dx_0) \bar{x}^j \bar{x}^k + \psi^i(\bar{x}) \tag{2.6}$$

where $\psi^i(\bar{x})$ are a set of functions, differentiable at least twice and vanishing together with first and second partial derivatives for $\bar{x}^i = 0$. From (2.6) we have

$$(\partial x^i / \partial \bar{x}^\beta)_0 = \delta_\beta{}^i; \qquad (\partial^2 x^i / \partial \bar{x}^\beta \partial \bar{x}^\gamma)_0 = -\Gamma^i{}_{\beta\gamma}(x_0, dx_0)$$

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* Cf. Berwald, *loc. cit.*, § 12, where he arrives at this definition of covariant differentiation from the notion of parallelism.

so that (1.11) become $\bar{\Gamma}^i{}_{\beta\gamma}(x_0, d\bar{x}_0) = 0$ and (1.13) become in geodesic coordinates

$$(2.7) \qquad (\bar{K}^i{}_{jkl})_0 = [(\partial\bar{\Gamma}^i{}_{jk}/\partial\bar{x}^l) - (\partial\bar{\Gamma}^i{}_{jl}/\partial\bar{x}^k)]_0.$$

From (2.5) in geodesic coordinates, we get by differentiation

$$(T^{(i)}{}_{(j),k,l})_0 = (\partial^2 T^{(i)}{}_{(j)}/\partial x^k \partial x^l)_0 - [T^{(i)}{}_{(j).h}(\partial\Gamma^h{}_{mk}/\partial x^l)\, dx^m]_0$$
$$+ \sum_{a=1}^{r} [T_{(j)}{}^{\cdots h \cdots}(\partial\Gamma^{i_a}_{hk}/\partial x^l)]_0 - \sum_{a=1}^{s} [T^{(i)}{}_{\ldots h \ldots}(\partial\Gamma^h{}_{j_a k}/\partial x^l)]_0.$$

When from this equation we subtract the result of interchanging k and l, we obtain, because of (2.7),

$$(2.8) \qquad T^{(i)}{}_{(j),k,l} - T^{(i)}{}_{(j),l,k} = - T^{(i)}{}_{(j).h} K^h{}_{mkl}\, dx^m$$
$$+ \sum_{a=1}^{r} T_{(j)}{}^{\cdots h \cdots} K^i{}_{hkl} - \sum_{a=1}^{s} T^{(i)}{}_{\ldots h \ldots} K^h{}_{j_a kl}.$$

Equations (2.8) being true at an arbitrary point for an arbitrary direction are true at any point in any direction. In a similar manner we get from (2.5) in geodesic coordinates

$$(T^{(i)}{}_{(j),k.l})_0 = [\partial^2 T^{(i)}{}_{(j)}/\partial x^k \partial(dx^l)]_0.$$
$$+ \sum_{a=1}^{r} (T_{(j)}{}^{\cdots h \cdots}\Gamma^{i_a}_{hkl} - \sum_{a=1}^{s} (T^{(i)}{}_{\ldots h \ldots}\Gamma^h{}_{j_a kl})_0$$

and

$$(T^{(i)}{}_{(j).l,k})_0 = [\partial^2 T^{(i)}{}_{(j)}/\partial x^k \partial(dx^l)]_0$$

Hence

$$(2.9) \qquad T^{(i)}{}_{(j),k.l} - T^{(i)}{}_{(j).l,k} = \sum_{a=1}^{r} T_{(j)}{}^{\cdots h \cdots}\Gamma^{i_a}_{hkl} \;)_0 - \sum_{a=1}^{s} T^{(i)}{}_{\ldots h \ldots}\Gamma^h{}_{j_a kl}$$

which are also true in general.*

From (2.7) or (1.11) we see that

$$(2.10) \qquad K^i{}_{jkl} + K^i{}_{jlk} = 0$$

and

$$(2.11) \qquad K^i{}_{jkl} + K^i{}_{klj} + K^i{}_{ljk} = 0.$$

From (1.13), in geodesic coordinates we have

$$(K^i{}_{jkl,m})_0 = \{(\partial^2\Gamma^i{}_{jk}/\partial x^l \partial x^m) - (\partial^2\Gamma^i{}_{jl}/\partial x^k \partial x^m)$$
$$- [\Gamma^i{}_{jkh}(\partial\Gamma^h{}_{pl}/\partial x^m) - \Gamma^i{}_{jlh}(\partial\Gamma^h{}_{pk}/\partial x^m]\, dx^p\}_0.$$

Interchanging k, l and m cyclically and adding the three equations, we find

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* Cf. Berwald, *loc. cit.*, p. 53.

$$(2.12)\qquad K^i{}_{jkl,m} + K^i{}_{jlm,k} + K^i{}_{jmk,l} + (K^h{}_{pkl}\Gamma^i{}_{jhm} + K^h{}_{plm}\Gamma^i{}_{jhk} + K^h{}_{pmk}\Gamma^i{}_{jhl})\,dx^p = 0.$$

Another identity that we shall need in our discussion and which can be obtained by the methods indicated above is

$$(2.13)\qquad \Gamma^i{}_{jkl,m} - \Gamma^i{}_{jkm,l} = K^i{}_{klm.j} = K^i{}_{jlm.k}.$$

From (2.13) it follows immediately that

$$(2.14)\qquad K^i{}_{jkl.m} + K^i{}_{jkm.l} + K^i{}_{jmk.l} = 0$$

and by contraction for i and j we have

$$(2.15)\qquad S_{kl.m} + S_{lm.k} + S_{mk.l} = 0.$$

S_{kl} being the alternating contracted curvature tensor $K^h{}_{hkl}$. If K_{jk} is the Ricci tensor $K^h{}_{jkh}$, it follows from (2.11) by contraction for i and l that

$$(2.16)\qquad K_{jk} - K_{kj} = -S_{jk}.$$

Hence K_{jk} will be symmetric if and only if $S_{jk} = 0$.

3. *Projective Invariants.* The equations of the paths of an affinely connected manifold with coefficients of connection defined by (1.3) may be written in the form *

$$(3.1)\qquad \dot{x}^i\,[(d\dot{x}^j/dt) + \Gamma^j{}_{kl}\dot{x}^k\dot{x}^l] - \dot{x}^j[(d\dot{x}^i/dt) + \Gamma^i{}_{kl}\dot{x}^k\dot{x}^l] = 0.$$

Equations (3.1) are invariant in form under arbitrary transformations of coordinates as well as under arbitrary transformations of the parameter t. We inquire whether it is possible for two different affine connections in the same continuum to have the same paths. Let $\Gamma^i{}_{jk}(x, \dot{x})$ and $\Gamma'^i{}_{jk}(x, \dot{x})$ be the two connections. When equations of the type (3.1) for each connection are subtracted one from the other we obtain

$$(3.2)\qquad (\Gamma'^i{}_{kl} - \Gamma^i{}_{kl})\dot{x}^j\dot{x}^k\dot{x}^l - (\Gamma'^j{}_{kl} - \Gamma^j{}_{kl})\dot{x}^i\dot{x}^k\dot{x}^l = 0.$$

From (1.11) it follows that $\Gamma'^i{}_{kl} - \Gamma^i{}_{kl}$ is a tensor, homogeneous of degree zero in $\dot{x}$ and symmetric in its subscripts; we designate it by $a^i{}_{kl}$ and write (3.2) in the form

$$(3.3)\qquad (a^i{}_{kl}\delta^j{}_h - a^j{}_{kl}\delta^i{}_h)\dot{x}^h\dot{x}^k\dot{x}^l = 0.$$

The quantities $a^i{}_{kl}\delta^j{}_h - a^j{}_{kl}\delta^i{}_h$ do not satisfy the second set of conditions of

* Cf. O. Veblen and T. Y. Thomas, *Transactions of the American Mathematical Society*, Vol. 25 (1923), pp. 557-560.

the lemma of § 1. We therefore differentiate (3.3) partially and successively with respect to $\dot{x}^p$, $\dot{x}^q$ and $\dot{x}^r$, the result being

$$(3.4)\qquad (a^i{}_{pq}\delta^j{}_r - a^j{}_{pq}\delta^i{}_r) + (a^i{}_{qr}\delta^j{}_p - a^j{}_{qr}\delta^i{}_p) + (a^i{}_{rp}\delta^j{}_q - a^j{}_{rp}\delta^i{}_q) + \dot{x}^j a^i{}_{pqr} - \dot{x}^i a^j{}_{pqr} = 0,$$

where

$$(3.5)\qquad a^i{}_{pqr} \equiv a^i{}_{pq.r} = a^i{}_{pr.q}.$$

When (3.4) are contracted for i and r we get

$$(3.6)\qquad a^i{}_{jk} = (\delta^i{}_j a^h{}_{hk} + \delta^i{}_k a^h{}_{jh} + \dot{x}^i a^h{}_{jkh})/(n+1).$$

From the definition of $a^i{}_{jk}$ and (1.3) it follows that $a^h{}_{hk}$ is a gradiant in the variables $\dot{x}$; hence we may let $a^h{}_{hk} = (n+1)\phi_{.k}$ so that (3.6) become

$$(3.7)\qquad a^i{}_{jk} = \delta^i{}_j\phi_{.k} + \delta^i{}_k\phi_{.j} + \dot{x}^i\phi_{.j.k},$$

ϕ being a homogeneous function of the first degree in $\dot{x}$. Conversely, if $\Gamma'^i{}_{jk} - \Gamma^i{}_{jk} = a^i{}_{jk}$ the latter being defined by (3.7) with ϕ entirely arbitrary—except as to homogeneity—the equations of the paths for the connection Γ' reduce to (3.1). Hence we have

A necessary and sufficient condition that two manifolds, with affine connections $\Gamma^i{}_{jk}(x, \dot{x})$ and $\Gamma'^i{}_{jk}(x, \dot{x})$ shall have the same paths is that

$$(3.8)\qquad \Gamma'^i{}_{jk} = \Gamma^i{}_{jk} + \delta^i{}_j\phi_{.k} + \delta^i{}_k\phi_{.j} + \dot{x}^i\phi_{.j.k},$$

ϕ being an arbitrary function homogeneous of the first degree in $\dot{x}$.

The connection Γ' in the above theorem is said to be obtained from the connection Γ by a *projective change* determined by the function ϕ. Or we may say, somewhat less definitely, that the connections Γ and Γ' are *projectively related.*

Contracting (3.8) for i and j we find $\phi_{.k} = (\Gamma'^h{}_{hk} - \Gamma^h{}_{hk})/(n+1)$ which when put into (3.8) give

$$\Pi'^i{}_{jk} = \Pi^i{}_{jk}$$

where

$$(3.9)\qquad \Pi^i{}_{jk} = \Gamma^i{}_{jk} - (1/n+1)\ (\delta^i{}_j\Gamma^h{}_{hk} + \delta^i{}_k\Gamma^h{}_{hj} + \dot{x}^i\Gamma^h{}_{hjk}).$$

Following T. Y. Thomas * we call the invariant defined by (3.9) the *projective connection,* $\Pi^i{}_{jk}$ being the *coefficients* of projective connection. They

* Cf. *Proceedings of the National Academy of Sciences,* Vol. 11 (1925), pp. 199-203. See also J. Douglas, *Annals of Mathematics,* 1928.

are obviously symmetric in the subscripts, are homogeneous of degree zero in $\dot{x}$ and have the property

$$\Pi^h{}_{hk} = 0. \tag{3.10}$$

By a *projective invariant* we shall mean one whose components are unaltered by a projective change of affine connection. It therefore follows immediately that

$$\begin{aligned} \Pi^i{}_{jkl} \equiv \Pi^i{}_{jk.l} = \Gamma^i{}_{jkl} \\ -(1/n+1)(\delta^i{}_j\Gamma^h{}_{hkl} + \delta^i{}_k\Gamma^h{}_{jhl} + \delta^i{}_l\Gamma^h{}_{jkh} + \dot{x}^i\Gamma^h{}_{jkl.h}) \end{aligned} \tag{3.11}$$

is a projective tensor invariant, which is symmetric in j, k, l.

When we form the expression for the curvature tensor of the connection Γ', a simple but rather long calculation gives

$$\begin{aligned} K'^i{}_{jkl} = K^i{}_{jkl} + \delta^i{}_j(\phi_{.k,l} - \phi_{.l,k}) + \delta^i{}_k(\phi_{.j,l} - \phi_{.j}\phi_{.l} - \phi\phi_{.j.l}) \\ -\delta^i{}_l(\phi_{.j,k} - \phi_{.j}\phi_{.k} - \phi\phi_{.j.k}) + \dot{x}^i(\phi_{.j.k,l} - \phi_{.j.l,k}). \end{aligned} \tag{3.12}$$

When (3.12) are contracted for i and l and for i and k we find

$$\begin{aligned} K'_{jk} = K_{jk} + (\phi_{.k,j} - \phi_{.j,k}) \\ -(n-1)(\phi_{.j,k} - \phi_{.j}\phi_{.k} - \phi\phi_{.j.k}) + \dot{x}^h\phi_{.j.k,h} \\ S'_{kl} = S_{kl} + (n+1)(\phi_{.k,l} - \phi_{.l,k}). \end{aligned} \tag{3.13}$$

In order to eliminate ϕ and its derivatives from (3.12) we have from the second of (3.13)

$$\phi_{.k,l} - \phi_{.l,k} = [(S'_{kl} - S_{kl})/(n+1)] \tag{3.14}$$

and therefore

$$\phi_{.k,l,j} - \phi_{.l,k.j} = [(S'_{kl.j} - S_{kl.j})/(n+1)]. \tag{3.15}$$

Now by means of (2.9) we get

$$\phi_{.k,l.j} = \phi_{.j.k,l} - \phi_{.h}\Gamma^h{}_{jkl}$$

this result and (3.15) give

$$\phi_{.j.k,l} - \phi_{.j.l,k} = [(S'_{kl.j} - S_{kl.j})/(n+1)]. \tag{3.16}$$

When (3.15) are multiplied by $\dot{x}^l$ and summed for l we obtain, since $\phi_{.j}$ is homogeneous of degree zero in $\dot{x}$,

$$\dot{x}^h\phi_{.j.k,h} = [\dot{x}^h/(n+1)]\,(S'_{kh.j} - S_{kh.j}). \tag{3.17}$$

We substitute (3.14) and (3.17) in the first of (3.13) and get

$$\begin{aligned} \phi_{.j,k} - \phi_{.j}\phi_{.k} - \phi\phi_{.j.k} = [\dot{x}^h/(n+1)]\,(S'_{kh.j} - S_{kh.j}) \\ -[1/(n-1)](K'_{jk} - K_{jk}) - [1/(n^2-1)](S'_{jk} - S_{jk}). \end{aligned} \tag{3.18}$$

When (3.14), (3.16) and (3.18) are substituted in (3.12) we obtain $W'^i{}_{jkl} = W^i{}_{jkl}$ where

$$(3.19)\quad \begin{aligned} W^i{}_{jkl} = K^i{}_{jkl} &+ [\delta_k{}^i/(n-1)]\, K_{jl} - [\delta_l{}^i/(n-1)]\, K_{jk} \\ &- [\delta_j{}^i/(n+1)]\, S_{kl} + [1/(n^2-1)]\, (\delta_k{}^i S_{jl} - \delta_l{}^i S_{jk}) \\ &- [\dot{x}^i/(n+1)]\, S_{kl.j} - [\dot{x}^h/(n^2-1)]\, (\delta^i{}_k S_{lh.j} - \delta^i{}_l S_{kh.j}), \end{aligned}$$

with a similar expression for $W'^i{}_{jkl}$. Following Weyl,* who discovered this tensor for affine connections of position, we call it the *projective curvature tensor,* since its components are unaltered by an arbitrary projective change of affine connection.

From (3.19) by means of (2.10), (2.11), (2.15) and (2.16) it follows that

$$(3.20)\quad \begin{aligned} &W^i{}_{jkl} = W^i{}_{jlk}, \\ &W^i{}_{jkl} + W^i{}_{klj} + W^i{}_{ljk} = 0, \\ &W^h{}_{hkl} = W^h{}_{klh} = 0. \end{aligned}$$

From (3.19) we have

$$\begin{aligned} W^i{}_{jkl.m} = K^i{}_{jkl.m} &+ [\delta^i{}_k/(n-1)]\, K_{jl.m} - [\delta^i{}_l/(n-1)]\, K_{jk.m} \\ &- [1/(n+1)]\, (\delta^i{}_j S_{kl.m} + \delta^i{}_m S_{kl.j}) - [\dot{x}^i/(n+1)]\, S_{kl.j.m} \\ &- [\dot{x}^h/(n^2-1)]\, (\delta^i{}_k S_{lh.j.m} - \delta^i{}_l S_{kh.j.m}) \\ &+ [\delta^i{}_k/(n^2-1)]\, (S_{jl.m} + S_{ml.j}) - [\delta^i{}_l/n^2-1)]\, (S_{jk.m} + S_{mk.j}). \end{aligned}$$

When (2.13) are contracted for i and m we find

$$K_{jk.l} = K_{lk.j};$$

hence $W^i{}_{jkl.m} = W^i{}_{mkl.j}$, since every group of terms in the expression for $W^i{}_{jkl.m}$ is symmetric in j and m. This fact and the second of (3.20) give

$$(3.21)\quad W^i{}_{jkl.m} + W^i{}_{jlm.k} + W^i{}_{jmk.l} = 0.$$

From (3.8) and (1.3) we get by contraction

$$(3.22)\quad \Gamma'^h{}_{hk} = \tfrac{1}{2} H^i{}_{.i.k} + (n+1)\phi_{.k}.$$

We choose for the components of the vector $\phi_{.k}$ in the given coordinate system the values $-[1/2(n+1)]H^i{}_{.i.k}$; then $\Gamma'^h{}_{hk} = 0$. An affine connection for which $\Gamma^h{}_{hk} = 0$, in some coordinate system, is said to be *normal.* From (3.22) we have the theorem

* Cf. *Göttinger Nachrichten* (1921), pp. 99-112.

*For a given affine connection there exists, in each coordinate system, a unique projectively related normal affine connection.**

From (1.13) it follows that for a normal affine connection $S_{kl} = 0$ and therefore the Ricci tensor K_{ij} is symmetric. From (3.9) we see that in this case $\Pi^i_{jk} = \Gamma^i_{jk}$ and (3.19) become

$$W^i_{jkl} = K^i_{jkl} + [\delta^i_k/(n-1)]\, K_{jl} - [\delta^i_l/(n-1)]\, K_{jk}. \tag{3.23}$$

4. *Transformation of Projective Connections. Projective Derivatives.* When we apply an arbitrary analytic transformation of coordinates

$$x^i = f^i(\bar{x}) \tag{4.1}$$

to a given affine connection, (1.11) gives

$$\bar{\Gamma}^\mu_{\mu\gamma} = \Gamma^h_{hk}(\partial x^k/\partial \bar{x}^\gamma) + (\partial \log \Delta/\partial \bar{x}^\gamma), \tag{4.2}$$

where $\Delta = |\,(\partial x/\partial \bar{x})\,|$ is the jacobian of (4.1). When this is applied to (3.9) we get the law of transformation of the components of projective connection in the form

$$\bar{\Pi}^\alpha_{\beta\gamma} \frac{\partial x^i}{\partial \bar{x}^\alpha} = \Pi^i_{jk} \frac{\partial x^j}{\partial \bar{x}^\beta} \frac{\partial x^k}{\partial \bar{x}^\gamma} - \frac{\partial x^i}{\partial \bar{x}^\beta} \frac{\partial \theta}{\partial \bar{x}^\gamma} - \frac{\partial x^i}{\partial \bar{x}^\gamma} \frac{\partial \theta}{\partial \bar{x}^\beta}, \tag{4.3}$$

where

$$\theta = [1/(n+1)] \log \Delta. \tag{4.4}$$

When we set up the conditions of integrability of (4.3) we arrive at

$$\bar{B}^\mu_{\alpha\beta\gamma} + \delta^\mu_\gamma \bar{c}_{\alpha\beta} - \delta^\mu_\beta \bar{c}_{\alpha\gamma} = B^i_{jkl} \frac{\partial x^j}{\partial \bar{x}^\alpha} \frac{\partial x^k}{\partial \bar{x}^\beta} \frac{\partial x^l}{\partial \bar{x}^\gamma} \frac{\partial \bar{x}^\mu}{\partial x^i}, \tag{4.5}$$

where

$$\begin{aligned} B^i_{jkl} = (\partial \Pi^i_{jk}/\partial x^l) - (\partial \Pi^i_{jl}/\partial x^k) + \Pi^h_{jk}\Pi^i_{hl} \\ - \Pi^h_{jl}\Pi^i_{hk} - (\Pi^i_{jkh}\Pi^h_{ml} - \Pi^i_{jlh}\Pi^h_{mk})\, dx^m \end{aligned} \tag{4.6}$$

and

$$\bar{c}_{\alpha\beta} = \bar{\Pi}^\epsilon_{\alpha\beta} \frac{\partial \theta}{\partial \bar{x}^\epsilon} + \frac{\partial \theta}{\partial \bar{x}^\alpha} \frac{\partial \theta}{\partial \bar{x}^\beta} - \frac{\partial^2 \theta}{\partial \bar{x}^\alpha \partial \bar{x}^\beta}. \tag{4.7}$$

When (4.5) are contracted for μ and γ, we get

$$\bar{c}_{\alpha\beta} = [1/(n-1)]\, (B^h_{jkh} \frac{\partial x^j}{\partial \bar{x}^\alpha} \frac{\partial x^k}{\partial \bar{x}^\beta} - B^\epsilon_{\alpha\beta\epsilon})$$

and these values put back into (4.5) give

$$W^\mu_{\alpha\beta\gamma} = W^i_{jkl} \frac{\partial x^j}{\partial \bar{x}^\alpha} \frac{\partial x^k}{\partial \bar{x}^\beta} \frac{\partial x^l}{\partial \bar{x}^\gamma} \frac{\partial \bar{x}^\mu}{\partial x^i},$$

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* Cf. Eisenhart, L. P., *Non-Riemannian Geometry*, § 37.

where

(4.8) $$W^i{}_{jkl} = B^i{}_{jkl} + [\delta^i{}_k/(n-1)]\, B^h{}_{jlh} - [\delta^i{}_l/(n-1)]\, B^h{}_{jkh};$$

direct computation shows that the tensor defined by (4.8) is identical with the Weyl tensor (3.19).*

We introduce the following notation:

(4.9) $$\begin{aligned} r_{jk} &= [1/(n-1)]\, B^h{}_{jkh} \\ &= [1/(n-1)]\, [(\partial\Pi^h{}_{jk}/\partial x^h) - \Pi^h{}_{jp}\Pi^p{}_{hk} - \Pi^p{}_{jkh}\Pi^h{}_{pm}dx^m], \\ \theta_l &= (\partial\theta/\partial x^l) \qquad\qquad u^i{}_j = (\partial x^i/\partial \bar{x}^j). \end{aligned}$$

Equations (4.7) may now be written as

(4.10) $$(\partial\theta_\alpha/\partial\bar{x}^\beta) = \bar{r}_{\alpha\beta} - r_{jk}u^j{}_\alpha u^k{}_\beta + \bar{\Pi}^\epsilon{}_{\alpha\beta}\theta_\epsilon + \theta_\alpha\theta_\beta.$$

When (4.10) are differentiated partially with respect to $d\bar{x}^\gamma$ and we let $r_{jk.l} = r_{jkl}$, we obtain

(4.11) $$\bar{r}_{\alpha\beta\gamma} = r_{jkl}u^j{}_\alpha u^k{}_\beta u^l{}_\gamma - \theta_\epsilon\bar{\Pi}^\epsilon{}_{\alpha\beta\gamma}.$$

From the first of (4.9) it follows that r_{jkl} is symmetric in all its subscripts. This invariant is not a tensor except when the projective connection is a function of position alone in which case $r_{jkl} = 0$. By using (4.3), (4.10) and (4.11) in the reduction, the other conditions of integrability of (4.10) can be written in the form

(4.12) $$\rho_{\alpha\beta\gamma} = \rho_{jkl}u^j{}_\alpha u^k{}_\beta u^l{}_\gamma - \theta_\epsilon\bar{W}^\epsilon{}_{\alpha\beta\gamma},$$

where

(4.13) $$\begin{aligned} \rho_{jkl} = (\partial r_{jk}/\partial x^l) - (\partial r_{jl}/\partial x^k) + \Pi^h{}_{jk}r_{hl} \\ - \Pi^h{}_{jl}r_{hk} - (r_{jkh}\Pi^h{}_{ml} - r_{jlh}\Pi^h{}_{mk})\, dx^m. \end{aligned}$$

The invariant ρ_{jkl} will be called the *projective covariant.*†

From a given projective invariant we can build a sequence of other projective invariants by projective differentiation.‡ We define the *projective derivative* of a projective invariant in the same way as the covariant derivative of a tensor was defined, [cf. (2.5)], except that the coefficients of affine connection are replaced by those of projective connection. It therefore follows that the identities existing between $K^i{}_{jkl}$ and $\Gamma^i{}_{jkl}$ are true for $B^i{}_{jkl}$ and

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* J. M. Thomas, *Proceedings of the National Academy of Sciences*, Vol. 11 (1925), pp. 207-209.

† Cf. O. Veblen and J. M. Thomas, *Annals of Mathematics*, Vol. 27 (1926), p. 287.

‡ This term has recently been adopted by O. Veblen to designate a different operation. Cf. § 9. It is used here in its original sense as defined below.

$\Pi^i{}_{jkl}$ provided covariant derivatives are replaced by projective derivatives the latter derivative being designated by a subscript preceded by a solidus. Thus (2.13) become

$$\Pi^i{}_{jkl/m} - \Pi^i{}_{jkm/l} = B^i{}_{klm.j} = B^i{}_{jlm.k}. \tag{4.14}$$

When this is contracted for i and m we obtain

$$r_{jkl} = [1/(n-1)]\, \Pi^h{}_{jkl/h} \tag{4.15}$$

and by means of (4.13) we can easily show that

$$\rho_{jkl} = r_{jk/l} - r_{jl/k}.$$

The identity analogous to (2.12) is

$$\begin{aligned} B^i{}_{jkl/m} &+ B^i{}_{jlm/k} + B^i{}_{jmk/l} \\ &+ (B^h{}_{pkl}\Pi^i{}_{jhm} + B^h{}_{plm}\Pi^i{}_{jhk} + B^h{}_{pmk}\Pi^i{}_{jhl})\, dx^p = 0. \end{aligned} \tag{4.16}$$

By means of (4.8) we find

$$B^i{}_{jkl/m} = W^i{}_{jkl/m} - \delta^i{}_k r_{jl/m} + \delta^i{}_l r_{jk/m}$$

and (4.16) then become

$$\begin{aligned} W^i{}_{jkl/m} + W^i{}_{jlm/k} + W^i{}_{jmk/l} &= \delta^i{}_k\rho_{jlm} + \delta^i{}_l\rho_{jmk} + \delta^i{}_m\rho_{jkl} \\ &\quad - (W^h{}_{pkl}\Pi^i{}_{jhm} + W^h{}_{plm}\Pi^i{}_{jhk} + W^h{}_{pmk}\Pi^i{}_{jhl})\, dx^p. \end{aligned} \tag{4.17}$$

By contraction for i and m the last equation becomes

$$W^h{}_{jkl/h} = (n-2)\rho_{jkl} - (W^h{}_{plm}\Pi^m{}_{jhk} - W^h{}_{pkm}\Pi^m{}_{jhl})\, dx^p. \tag{4.18}$$

Since when $n = 2$ $B^i{}_{jkl}$ has only three independent components r_{11}, r_{12} and r_{22}, it is easy to show that in that case the Weyl tensor vanishes identically. By (4.12) ρ_{jkl} becomes a tensor and (4.18) are satisfied independently of the value of ρ_{jkl}. We shall find that the projective geometry of a manifold of more than two dimensions is completely characterized by the tensors $W^i{}_{jkl}$ and $\Pi^i{}_{jkl}$ and their projective and partial derivatives with respect to dx. For $n = 2$ ρ_{jkl} and $\Pi^i{}_{jkl}$ with their derivatives play the same rôle.

5. *Collineations.* Let $\bar{x}^i = \phi^i(x)$ be a point transformation which carries the point $(x^1, \cdots, x^n)$ into the point of coordinates $(\bar{x}^1, \cdots, \bar{x}^n)$. Let $x^i = f^i(t)$ be the finite equations of a path C, the parameter t being affine, that is, the equations of C satisfying

$$\frac{d^2x^i}{dt} + \Gamma^i{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0. \tag{5.1}$$

The transformation $\bar{x}^i = \phi^i(x)$ will carry C into a curve $\bar{C}$ whose finite equations are of the form $\bar{x}^i = \psi^i(t)$. This transformation will be called a *projective collineation* if it carries each path C into a path $\bar{C}$. The collineation will be called affine if when C satisfies equations (5.1) so does $\bar{C}$.

In the present paragraph we shall confine ourselves to projective collineations. Since in this case the parameter on the path is not necessarily preserved it will be convenient to take the equations of the paths in the form (3.1) which are unchanged by arbitrary transformations of parameter and may therefore be written as

$$(5.2) \qquad (d^2x^i + \Gamma^i_{jk}dx^jdx^k)\,dx^h - (d^2x^h + \Gamma^h_{jk}dx^jdx^k)\,dx^i = 0.$$

Moreover, we shall consider only finite continuous groups of projective collineations and our problem will be to find the necessary and sufficient conditions that a space of paths must satisfy in order to admit such a group.

We consider the infinitesimal transformations of a one parameter group which may be written as

$$(5.3) \qquad \bar{x}^i = x^i + \xi^i(x)\delta u$$

where $\xi^i(x)$ are the components of a contravariant vector and δu is a differential of the parameter of the group. By differentiation we obtain from (5.3)

$$(5.4) \qquad d\bar{x}^i = dx^i + (\partial\xi^i/\partial x^j)\,dx^j\delta u$$

and

$$(5.5) \qquad d^2\bar{x}^i = d^2x^i + [(\partial^2\xi^i/\partial x^j\partial x^k)\,dx^jdx^k + (\partial\xi^i/\partial x^j)\,d^2x^j]\,\delta u.$$

If (5.3) is to define an infinitesimal collineation $\Gamma^i_{jk}(\bar{x}, d\bar{x})$ must be the same functions of $\bar{x}$ and $d\bar{x}$ as $\Gamma^i_{jk}(x, dx)$ are of x and dx. Therefore expanding into a Taylor series we have

$$(5.6) \qquad \Gamma^i_{jk}(\bar{x}, d\bar{x}) = \Gamma^i_{jk}(x, dx) + [(\partial\Gamma^i_{jk}/\partial x^l)\xi^l + \Gamma^i_{jkl}(\partial\xi^l/\partial x^m)\,dx^m]\delta u,$$

higher powers of δu than the first being neglected throughout the problem. When we write (5.2) in terms of $\bar{x}$ and apply the transformations (5.3), (5.4), (5.5), (5.6) and use the fact that the transformation is a collineation the transformed equations reduce to

$$\begin{aligned}
&[(\partial^2\xi^i/\partial x^j\partial x^k) - \Gamma^a_{jk}(\partial\xi^i/\partial x^a) + \Gamma^i_{ja}(\partial\xi^a/\partial x^k) + \Gamma^i_{ak}(\partial\xi^a/\partial x^j) \\
&\quad + \xi^a(\partial\Gamma^i_{jk}/\partial x^a) + \Gamma^i_{jkl}(\partial\xi^l/\partial x^a)\,dx^a]\,dx^hdx^jdx^k \\
&\quad - [(\partial^2\xi^h/\partial x^j\partial x^k) - \Gamma^a_{jk}(\partial\xi^h/\partial x^a) + \Gamma^h_{ja}(\partial\xi^a/\partial x^k + \Gamma^h_{ak}(\partial\xi^a/\partial x^j) \\
&\quad + \xi^a(\partial\Gamma^h_{jk}/\partial x^a) + \Gamma^h_{jkl}(\partial\xi^l/\partial x^a)\,dx^a]\,dx^idx^jdx^k = 0,
\end{aligned}$$

where the arbitrary δu has been dropped.

If we compare the first parenthesis in the above equations with $\xi^i{}_{,j,k}$ whose value is given by

$$(5.61)\qquad \xi^i{}_{,j,k} = (\partial^2\xi^i/\partial x^j\partial x^k) - \Gamma^a{}_{jk}(\partial\xi^i/\partial x^a) + \Gamma^i{}_{ja}(\partial\xi^a/\partial x^k) + \Gamma^i{}_{ak}(\partial\xi^a/\partial x^j) + \xi^a(\partial\Gamma^i{}_{ja}/\partial x^k) - \xi^a\Gamma^i{}_{ja\beta}\Gamma^\beta{}_{km}dx^m + \xi^a\Gamma^\beta{}_{ja}\Gamma^i{}_{\beta k} - \xi^a\Gamma^\beta{}_{jk}\Gamma^i{}_{\beta a},$$

we shall find that the equations of condition reduce to

$$(5.7)\qquad a^i{}_{jk}dx^h dx^j dx^k - a^h{}_{jk}dx^i dx^j dx^k = 0,$$

where

$$(5.8)\qquad a^i{}_{jk} = \xi^i{}_{,j,k} + \xi^l K^i{}_{jkl} + \Gamma^i{}_{jkl}\xi^l{}_{,m}dx^m.$$

Because ξ is a vector of position alone and in virtue of (2.8) and (2.11), $a^i{}_{jk}$ is a tensor, symmetric in its subscripts, homogeneous of degree zero in dx. By using (2.9) twice and (2.13) we find

$$(5.9)\qquad a^i{}_{jk.l} = \xi^h\Gamma^i{}_{jkl,h} + \xi^h{}_{,m}\Gamma^i{}_{jkl.h}dx^m - \xi^i{}_{,h}\Gamma^h{}_{jkl} + \xi^h{}_{,j}\Gamma^i{}_{hkl} + \xi^h{}_{,k}\Gamma^i{}_{jhl} + \xi^h{}_{,l}\Gamma^i{}_{jkh},$$

from which it is obvious that $a^i{}_{jkl} = a^i{}_{jk.l}$, is symmetric in its subscripts. Hence if we write (5.7) in the form (3.3), it follows from (3.7) that (5.7) may be replaced by equations of the form

$$(5.10)\qquad \xi^i{}_{,j,k} + \xi^l K^i{}_{jkl} + \Gamma^i{}_{jkl}\xi^l{}_{,m}dx^m = \delta^i{}_j\phi_{.k} + \delta^i{}_k\phi_{.j} + dx^i\cdot\phi_{.j.k}$$

where ϕ is to be determined.

The above equations are a necessary condition that the vector ξ must satisfy in order to define an infinitesimal projective collineation. To show that it is sufficient we write (5.10) in the form

$$(5.11)\qquad (\partial^2\xi^i/\partial x^j\partial x^k) - \Gamma^m{}_{jk}(\partial\xi^i/\partial x^m) + \Gamma^i{}_{jm}(\partial\xi^m/\partial x^k) + \Gamma^i{}_{mk}(\partial\xi^m/\partial x^j) + \xi^m(\partial\Gamma^i{}_{jk}/\partial x^m) + \Gamma^i{}_{jkl}(\partial\xi^l/\partial x^m)dx^m = \delta^i{}_j\phi_{.k} + \delta^i{}_k\phi_{.j} + dx^i\phi_{.j.k}.$$

Contracting for i and j we shall find

$$\phi_{.k} = [1/(n+1)]\,[(\partial^2\xi^h/\partial x^h\partial x^k) + \Gamma^h{}_{hm}(\partial\xi^m/\partial x^k) + \xi^m(\partial\Gamma^h{}_{hk}/\partial x^m) + \Gamma^h{}_{hkl}(\partial\xi^l/\partial x^m)dx^m]$$

and when we evaluate $\phi_{.j.k}$ and put these results in (5.11) they become

(5.12)

$$(\partial^2\xi^i/\partial x^j\partial x^k) - \Pi^m{}_{jk}(\partial\xi^i/\partial x^m) + \Pi^i{}_{jm}(\partial\xi^m/\partial x^k) + \Pi^i{}_{mk}(\partial\xi^m/\partial x^j) + \xi^m(\partial\Pi^i{}_{jk}/\partial x^m) + \Pi^i{}_{jkl}(\partial\xi^l/\partial x^m)dx^m = [1/(n+1)]\,[\delta^i{}_j(\partial^2\xi^h/\partial x^h\partial x^k) + \delta^i{}_k(\partial^2\xi^h/\partial x^h\partial x^j)].$$

Now suppose that (5.12) admit a solution ξ^i. By a suitable transformation of coordinates * we can normalize the vector ξ^i so that its components are δ^i_1. Equations (5.10) being in tensor form remain in the same form and so do (5.12), the latter reducing in this coordinate system to

$$(\partial \Pi^i{}_{jk}/\partial x^1) = 0. \tag{5.13}$$

The finite transformations determined by the above vector are

$$\bar{x}^i = x^i + a\delta^i{}_1 \tag{5.14}$$

and since the jacobian $\Delta = 1$, $\bar{\Pi}^i{}_{jk} = \Pi^i{}_{jk}$, cf. (4.3). Therefore the transformations (5.14) carry paths into paths. We therefore have

A necessary and sufficient condition that a vector ξ^i must satisfy in order to define an infinitesimal projective collineation is that it be a solution of (5.12).

From (5.13) we can conclude

The most general projective connection admitting an infinitesimal projective collineation may be obtained by choosing for the coefficients of projective connection functions of $n-1$ of the coordinates.

From (5.14) we have

A manifold admitting an infinitesimal projective collineation admits the finite continuous group G_1 of projective collineations.

Returning to (5.12) we see that when the left-hand side of (5.12) is compared with $\xi^i{}_{/j/k}$, [cf. (5.61)] with Γ's replaced by Π's, we obtain

$$\xi^i{}_{/j/k} + \xi^l B^i{}_{jkl} + \xi^l{}_{/m} dx^m \Pi^i{}_{jkl} = \delta^i{}_j \psi_{/k} + \delta^i{}_k \psi_{/j}, \tag{5.15}$$

where,

$$\psi_{/j} = [1/(n+1)]\, \xi^h{}_{/h/j} \tag{5.16}$$

and it is to be noted that $\psi_{/j.k} = 0$.

To find the conditions of integrability of (5.15) we have

$$\xi^i{}_{/j/k/l} = \delta^i{}_j \psi_{/k/l} + \delta^i{}_k \psi_{/j/l} - dx^m \psi_{/m} \Pi^i{}_{jkl} + dx^m \xi^h \Pi^i{}_{jkp} B^p{}_{mlh}$$
$$- \xi^h{}_{/m} dx^m \Pi^i{}_{jkh/l} - \xi^h{}_{/l} B^i{}_{jkh} - \xi^h B^i{}_{jkh/l}.$$

If from this equation we subtract the one obtained from it by the interchange of k and l, make use of the fact that

$$\xi^i{}_{/j.p} = \xi^h \Pi^i{}_{jhp}, \tag{5.17}$$

of (2.8) and (2.11) in projective form and of (4.14) and (4.16), we shall have as the result

---

* Cf. L. P. Eisenhart, "Riemannian Geometry," p. 5.

$$(5.18) \qquad \xi^h B^i{}_{jkl/h} + \xi^h{}_{/m} B^i{}_{jkl.h} dx^m - \xi^i{}_{/h} B^h{}_{jkl} + \xi^h{}_{/j} B^i{}_{hkl} + \xi^h{}_{/k} B^h{}_{jhl} + \xi^h{}_{/l} B^i{}_{jkh} = \delta^i{}_k \psi_{/j/l}.$$

The conditions of integrability of (5.15) with regard to dx are obtainable by the partial differentiation of (5.15). We note that by (2.9), written in projective form of course, we have

$$\xi^i{}_{/j/k.l} = \xi^i{}_{/j.l/k} + \xi^h{}_{/j}\Pi^i{}_{hkl} - \xi^i{}_{/h}\Pi^h{}_{jkl}$$

and by the use of (5.17) this becomes

$$(5.19) \qquad \xi^i{}_{/j/k.l} = \xi^h\Pi^i{}_{jlh/k} + \xi^h{}_{/k}\Pi^i{}_{jhl} + \xi^h{}_{/j}\Pi^i{}_{hkl} - \xi^i{}_{/h}\Pi^h{}_{jkl}.$$

We therefore have by (4.14), (cf. remark after (5.16)),

$$(5.20) \qquad \xi^h\Pi^i{}_{jkl/h} + \xi^h{}_{/m}dx^m\Pi^i{}_{jkl.h} - \xi^i{}_{/h}\Pi^h{}_{jkl} + \xi^h{}_{/j}\Pi^i{}_{hkl} + \xi^h{}_{/k}\Pi^i{}_{jhl} + \xi^h{}_{/l}\Pi^i{}_{jkh} = 0.$$

By contraction in (5.18) for i and l we obtain

$$(5.21) \qquad \psi_{/j/k} = -(\xi^h r_{jk/h} + \xi^h{}_{/m}dx^m r_{jkh} + \xi^h{}_{/j} r_{hk} + \xi^h{}_{/k} r_{hj}).$$

When the last results are substituted in (5.18) we get because of (4.8) and the first of (4.9)

$$(5.22) \qquad \xi^h W^i{}_{jkl/h} + \xi^h{}_{/m}dx^m W^i{}_{jkl.h} - \xi^i{}_{/h} W^h{}_{jkl} + \xi^h{}_{/j} W^i{}_{hkl} + \xi^h{}_{/k} W^i{}_{jhl} + \xi^h{}_{/l} W^i{}_{jkh} = 0.$$

That the above equations are invariant under an arbitrary projective change of affine connection is evident, since each term is independent of this change. When we express each projective derivative in terms of covariant derivatives, the form of (5.22) remains unaltered.* Hence (5.22) are tensor equations, i. e., they are invariant under an arbitrary analytic transformation of coordinates.

We must now examine the conditions of integrability of (5.21). To this end we differentiate (5.21) partially with respect to dx^l and use the fact that $\psi_{/j.l} = 0$, getting

$$(5.23) \qquad \psi_{/h}\Pi^h{}_{jkl} = \xi^h r_{jkl/h} + \xi^h{}_{/j} r_{hkl} + \xi^h{}_{/k} r_{jhl} + \xi^h{}_{/l} r_{jkh}.$$

If we differentiate (5.20) projectively and use (2.8) and (2.9) in projective form we shall get

---

* Cf. M. S. Knebelman, *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), p. 397.

$$
\begin{aligned}
(5.24) \qquad & \xi^h \Pi^i{}_{jkl/m/h} + \xi^h{}_{/s} dx^s \Pi^i{}_{jkl/m.h} - \xi^i{}_{/h} \Pi^h{}_{jkl/m} + \xi^h{}_{/j} \Pi^i{}_{hkl/m} \\
& + \xi^h{}_{/k} \Pi^i{}_{jhl/m} + \xi^h{}_{/l} \Pi^i{}_{jkh/m} + \xi^h{}_{/m} \Pi^i{}_{jkl/h} \\
& + \Pi^i{}_{jkl} \psi_{/m} + \psi_{/s} dx^s \Pi^i{}_{jkl.m} - \delta^i{}_m \Pi^h{}_{jkl} \psi_{/h} \\
& + \Pi^i{}_{mkl} \psi_{/j} + \Pi^i{}_{jml} \psi_{/k} + \Pi^i{}_{jkm} \psi_{/l} = 0.
\end{aligned}
$$

When (5.24) are contracted for i and m and use is made of (4.15) we obtain (5.23). Hence in studying the conditions that the vector ξ^i must satisfy we need not consider the equations arising from (5.21) by partial differentiation.

We now take the projective derivative of (5.21) obtaining after reduction by means of (5.15)

$$
\begin{aligned}
\psi_{/h} B^h{}_{jkl} = \xi^h \rho_{jkl/h} + \xi^h{}_{/m} dx^m \rho_{jkl.h} + \xi^h{}_{/j} \rho_{hkl} \\
+ \xi^h{}_{/k} \rho_{jhl} + \xi^h{}_{/l} \rho_{jkh} + r_{jk} \psi_{/l} - r_{jl} \psi_{/k},
\end{aligned}
$$

which may be written as, cf. (4.8), (4.9),

$$
(5.25) \qquad \psi_{/h} W^h{}_{jkl} = \xi^h \rho_{jkl/h} + \xi^h{}_{/m} dx^m \rho_{jkl.h} + \xi^h{}_{/j} \rho_{hkl} + \xi^h{}_{/k} \rho_{jhl} + \xi^h{}_{/l} \rho_{jkh}.
$$

Projective differentiation of (5.22) gives

$$
\begin{aligned}
(5.26) \qquad & \xi^h W^i{}_{jkl/m/s} + \xi^s{}_{/v} dx^v W^i{}_{jkl/m.s} - \xi^i{}_{/h} W^h{}_{jkl/m} + \xi^h{}_{/k} W^i{}_{jhl/m} \\
& + \xi^h{}_{/l} W^i{}_{jkh/m} + \xi^h{}_{/m} W^i{}_{jkl/h} + 2 W^i{}_{jkl} \psi_{/m} \\
& - \delta^i{}_m W^h{}_{jkl} \psi_{/h} + W^i{}_{mkl} \psi_{/j} + W^i{}_{jml} \psi_{/k} + W^i{}_{jkm} \psi_{/l} = 0.
\end{aligned}
$$

By contraction for i and m we get from (5.26)

$$
\begin{aligned}
(5.27) \qquad (n-2) W^h{}_{jkl} \psi_{/h} = \xi^s W^h{}_{jkl/h/s} + \xi^s{}_{/v} dx^v W^h{}_{skl/h} \\
+ \xi^s{}_{/k} W^h{}_{jsl/h} + \xi^s{}_{/l} W^h{}_{jks/h}.
\end{aligned}
$$

When $n > 2$ (5.27) can be reduced by means of (4.18), (5.20) and (5.22) to (5.25) thus showing that in this case the conditions of integrability of (5.21) are consequences of (5.20) and (5.22). In the case $n = 2$ (5.22) vanish identically and their place is taken by (5.25).

6. *Spaces of paths admitting an r-parameter group of projective collineations.* If a space is to admit an r-parameter group G_r of projective collineations equations (5.15) must admit r linearly independent solutions $\xi^i{}_{(1)}, \cdots, \xi^i{}_{(r)}$, each vector $\xi^i{}_{(a)}$ being a vector of position. The question of the existence and of the number of solutions of a system of partial differential equations is answered by a well-known theorem on mixed systems.*

* L. Bianchi, *Teoria dei Gruppi Continui*, Pisa (1918), Chap. I; O. Veblen, "Invariants of Quadratic Differential Forms," *Cambridge Tract*, No. 24, pp. 73-76.

The theorem has to be somewhat modified in order to apply to equations (5.15). In the first place (5.15) are a pure system, that is, there are no conditions of the form $f(\xi, x) = 0$ imposed on the solutions and secondly it is obviously only a question of convenience whether we write our equations and their conditions of integrability in terms of the usual partial derivatives, covariant derivatives or projective derivatives.

Let " $;\alpha$ " denote projective differentiation when $\alpha = 1, \cdots, n$ and partial differentiation with respect x^a when $\alpha = n+1, \cdots, 2n$ where x^{n+k} is used in place of dx^k. We may then write (5.15) as

$$Z^\mu{}_{;\alpha} = f^\mu{}_\alpha(x, Z) \begin{pmatrix} \mu = 1, \cdots, n^2 + 2n \\ \alpha = 1, \ldots, 2n \end{pmatrix}, \tag{6.1}$$

Z^μ being the dependent variables ξ^i, $\xi^i/_j$ and $\psi/_j$. In order to obtain the conditions of integrability we have (2.5) and (2.9) written in terms of projective derivatives which we can combine into

$$Z^\mu{}_{;\alpha;\beta} - Z^\mu{}_{;\beta;\alpha} = A^{\mu\gamma}{}_{\lambda\alpha\beta} Z^\lambda{}_{;\gamma}. \tag{6.2}$$

Thus by means of (6.2) we get from (6.1) a system of equations between x and Z which we denote by $F^{(1)}(x, Z)$. Differentiating each of the independent equations of the set $F^{(1)}$ and eliminating $Z^\mu{}_{;\alpha}$ by means of (6.1) we arrive at a set $F^{(2)}(x, Z)$ this process being repeated indefinitely. The theorem mentioned at beginning of this paragraph may be stated thus:

A necessary and sufficient condition that the functions $f^\mu{}_\alpha$ must satisfy in order that (6.2) admit a solution is that the sets of equations $F^{(1)}$, $F^{(2)}$, $F^{(3)} \cdots$ shall be compatible for the determination of the unknown quantities Z.

In the particular problem before us $f^\mu{}_\alpha(x, Z)$ for the system (5.15) are linear and homogeneous in Z. Hence the above theorem may be stated in a more explicit form since in this case the sets of equations $F^{(1)}$, $F^{(2)}, \cdots$ are also linear homogeneous in Z. The theorem is

A necessary and sufficient condition that the sets $F^{(1)}$, $F^{(2)}, \cdots$ must satisfy in order to be compatible for the determination of the Z's is that there exist a positive integer $N (\geq 1)$ such that the matrix of the equations of the sets $F^{(1)}, \cdots, F^{(N)}$ and that of the equations of the sets $F^{(1)}, \cdots, F^{(N+1)}$ shall have the same rank. If M is the number of dependent functions Z

and $M - r$ $(1 \leq r \leq M)$* *is the common rank of the above matrices, the number of linearly independent solutions is* r.

If $r = M$ the conditions of integrability are satisfied identically and the equations (6.1) are then said to be completely integrable.

To apply the above results to the problem of projective collineations, we take for Z^a the functions ξ^i, $\xi^i{}_{/j}$, $\psi_{/j}$ and for the operation " $;i$ " we choose $;i = /i$ $(i = 1, \cdots, n)$ and $;i = .i$ $(i = n+1, \cdots, 2n)$, the independent variables being so chosen that $x^i = x^i$, $x^{n+i} = dx^i$ $(i = 1, \cdots, n)$. The equations corresponding to (6.1) are then

$$(6.3) \qquad \xi^i{}_{.k} = 0, \quad (\psi_{/j})_{.k} = 0, \quad (\xi^i{}_{/j})_{.k} = \xi^h \Pi^i{}_{hjk}, \quad (\xi^i)_{/k} = \xi^i{}_{/k}$$

$$(5.21) \qquad (\psi_{/j})_{/k} = -(\xi^h r_{jk/h} + \xi^h{}_{/m} dx^m r_{jkh} + \xi^h{}_{/j} r_{hk} + \xi^h{}_{/k} r_{hj})$$

and

$$(6.4) \qquad (\xi^i{}_{/j})_{/k} = -(\xi^h B^i{}_{jkh} + \xi^h{}_{/m} dx^m \Pi^i{}_{jkh} - \delta^i{}_j \psi_{/k} - \delta^i{}_k \psi_{/j}).$$

The conditions of integrability of (6.3) are satisfied because of (6.3) and (2.9) and the conditions of integrability of (5.21) and (6.4) are (5.20) and (5.22). We therefore have

A necessary and sufficient condition that a projective connection must satisfy in order to admit r linearly independent one parameter finite continuous groups of projective collineations $(1 \leq r \leq n^2 + 2n)$ *is that there exist a positive integer N such that the sets of equations* $F^{(1)}, \cdots, F^{(N)}$ *and the sets* $F^{(1)}, \cdots, F^{(N+1)}$, *arising from the equations* (5.20) *and* (5.22), *shall be of rank* $n^2 + 2n - r$.

If equations (5.15) are to be completely integrable, (5.20) and (5.22) must vanish identically. If we write these equations in the form

$$(6.5) \qquad \begin{aligned} &\xi^h \Pi^i{}_{jkl/h} + \xi^h{}_{/m}(dx^m \Pi^i{}_{jkl.h} - \delta^i{}_h \Pi^m{}_{jkl} \\ &\qquad + \delta^m{}_j \Pi^i{}_{hkl} + \delta^m{}_k \Pi^i{}_{jhl} + \delta^m{}_l \Pi^i{}_{jkh}) = 0 \\ &\xi^h W^i{}_{jkl/h} + \xi^h{}_{/m}(dx^m W^i{}_{jkl.h} - \delta^i{}_h W^m{}_{jkl} \\ &\qquad + \delta^m{}_j W^i{}_{hkl} + \delta^m{}_k W^i{}_{jhl} + \delta^m{}_l W^i{}_{jkh}) = 0, \end{aligned}$$

it follows that

$$(6.6) \qquad \Pi^i{}_{jkl/h} = W^i{}_{jkl/h} = 0$$

and each parenthesis in (6.5) must vanish.

When these last two quantities are contracted for m and h we get

---

* We do not consider the case $r = 0$ since the equations are linear and homogeneous in the Z's. In the problem of equivalence $r = 0$ implies equivalence under a particular transformation while $r > 0$ implies equivalence under a finite continuous group of transformations. Cf. Eisenhart, *Non-Riemannian Geometry*, pp. 16-17.

$$\Pi^i{}_{jkl} = 0, \qquad 2W^i{}_{jkl} = 0. \tag{6.7}$$

From the first of (6.7) and the definition of $\Pi^i{}_{jkl}$, Cf. (3.11), it follows that the projective connection is a function of position and the second set of equations (6.7) for $n > 2$ are a necessary and sufficient condition for the space of paths to be projectively flat.* In case $n = 2$ the process applied above to (5.23) and (5.25), yields $\rho_{ijk} = 0$ and $r_{ijk} = 0$ which implies that the two-space is projectively flat. Hence

A necessary and sufficient condition that a projective connection must satisfy in order to admit $n^2 + 2n$ linearly independent projective collineations is that it be projectively flat.

7. *Group Property of Projective Collineations.* In § 5 it was proved that when a space of paths admits an infinitesimal collineation it admits the one parameter group G_1 generated by it. We shall now show that

If a space of paths admits r linearly independent collineations it admits the r-parameter group G_r generated by them.

For let ξ^i and η^i be any two of the r linearly independent solutions that equations (5.15) are assumed to admit and let $X \equiv \xi^i(\partial/\partial x^i)$, $Y \equiv \eta^i(\partial/\partial x^i)$ be the corresponding generators. In order to prove the above theorem it is necessary and sufficient to show that the Poisson operator (X, Y) is expressible linearly with constant coefficients in terms of the r generators. Let $(X, Y) = \lambda^i(\partial/\partial x^i)$. Then

$$\begin{aligned}(X, Y)f &= (XY - YX)f \\ &= [\xi^i(\partial\eta^j/\partial x^i) - \eta^i(\partial\xi^j/\partial x^i)]\,(\partial f/\partial x^j) = \lambda^j(\partial f/\partial x^j)\end{aligned} \tag{7.1}$$

and since (7.1) must hold for arbitrary functions we have

$$\lambda^i = \xi^k(\partial\eta^i/\partial x^k) - \eta^k(\partial\xi^i/\partial x^k)$$

or better

$$\lambda^i = \xi^k\eta^i{}_{/k} - \eta^k\xi^i{}_{/k}. \tag{7.2}$$

From (7.2) we obtain by projective differentiation

$$\lambda^i{}_{/j} = \xi^h\eta^i{}_{/h/j} - \eta^h\xi^i{}_{/h/j} + \xi^h{}_{/j}\eta^i{}_{/h} - \eta^h{}_{/j}\xi^i{}_{/h}$$

and since ξ^i and η^i satisfy (5.15), we get by means of these equations

$$\begin{aligned}\lambda^i{}_{/j} = {}& \xi^h{}_{/j}\eta^i{}_{/h} - \eta^h{}_{/j}\xi^i{}_{/h} + \xi^i\phi_{/j} - \eta^i\psi_{/j} + \delta^i{}_j(\xi^h\phi_{/h} - \eta^h\psi_{/h}) \\ & + \xi^h\eta^l B^i{}_{jlh} - \Pi^i{}_{jlh}(\xi^h\eta^l{}_{/m} - \eta^h\xi^l{}_{/m})\,dx^m,\end{aligned} \tag{7.3}$$

* Cf. O. Veblen and J. M. Thomas, *Annals of Mathematics*, Vol. 27 (1926), p. 293-, § 11.

where in accordance with (5.16)

$$\psi_{/j} = [1/(n+1)]\,\xi^h{}_{/h/j}, \qquad \phi_{/j} = [1/(n+1)]\eta^h{}_{/h/j}.$$

When (7.3) are differentiated projectively and second derivatives of ξ^i and η^i are eliminated by means of equations (5.15) we obtain

$$\begin{aligned}\lambda^i{}_{/j/k} = {} & \xi^i{}_{/h}(\eta^l B^h{}_{jkl} + \eta^l{}_{/m}dx^m\Pi^h{}_{jkl}) + \xi^h{}_{/j}(\eta^l B^i{}_{hkl} + \eta^l{}_{/m}dx^m\Pi^i{}_{hkl}) \\ & - \eta^i{}_{/h}(\xi^l B^h{}_{jkl} + \xi^l{}_{/m}dx^m\Pi^h{}_{jkl}) - \eta^h{}_{/j}(\xi^l B^i{}_{hkl} + \xi^l{}_{/m}dx^m\Pi^i{}_{hkl}) \\ & + \xi^h\eta^l B^i{}_{jhl/k} + B^i{}_{jlh}(\eta^l\xi^h{}_{/k} - \xi^l\eta^h{}_{/k}) - (\xi^h\eta^l{}_{/m} - \eta^h\xi^l{}_{/m})\Pi^i{}_{jlh/k}dx^m \\ & - \Pi^i{}_{jlh}\,[\xi^h{}_{/k}\eta^l{}_{/m} - \eta^h{}_{/k}\xi^l{}_{/m} + B^l{}_{mkp}(\eta^h\xi^p - \eta^p\xi^h)]\,dx^m \\ & + \xi^i\phi_{/j/k} - \eta^i\psi_{/j/k} + \delta^i{}_j(\xi^h\phi_{/h} - \eta^h\psi_{/h})_{/k}.\end{aligned}$$

When the values of $\Pi^i{}_{jkl}\lambda^l{}_{/m}dx^m$ and of $\lambda^h B^i{}_{jhk}$ computed by means of (7.2) and (7.3), are added to the above equations the result is reducible to six groups of terms; the first two of the form (5.18), the second two of the form (5.19) the remaining two being $\delta^i{}_j(\xi^h\phi_{/h} - \eta^h\psi_{/h})_{/k}$ and $\delta^i{}_k(\xi^h\phi_{/h} - \eta^h\psi_{/h})_{/j}$. That is, we find

$$\begin{aligned}(7.4)\qquad \lambda^i{}_{/j/k} + \Pi^i{}_{jkl}\lambda^l{}_{/m}dx^m + \lambda^l B^i{}_{jkl} \\ = \delta^i{}_k(\xi^h\phi_{/h} - \eta^h\psi_{/h})_{/j} + \delta^i{}_j(\xi^h\phi_{/h} - \eta^h\psi_{/h})_{/k}.\end{aligned}$$

When (7.3) are contrasted for i and j, the result is

$$\xi^h\phi_{/h} - \eta^h\psi_{/h} = [1/(n+1)]\,\lambda^h{}_{/h}$$

so that the right hand side of (7.4) can be written as $[1/(n+1)]\,(\delta^i{}_j\lambda^h{}_{/h/k} + \delta^i{}_k\lambda^h{}_{/h/j})$, showing that (7.4) are of the form (5.15). That is, λ^i is therefore a solution of (5.15) which are homogeneous of the first degree; this implies that if $\xi^i{}_{(a)}$ $(\alpha = 1, \cdots, r)$ are the r linearly independent solutions of (5.15), $\lambda^i = a^a\xi^i{}_{(a)}$, where a^a are a set of constants. Hence if $X_a \equiv \xi^i{}_{(a)}(\partial/\partial x^i)$ we have

$$(X_a, X_\beta)f = c^\gamma{}_{a\beta}X_\gamma f$$

$c^\gamma{}_{a\beta}$ being the constants of composition of the group.

8. *Affine Collineations.* By an affine collineation we shall understand a point transformation carrying paths into paths and preserving the affine parameter. That is, if a path is given in terms of an affine parameter, it is carried by an affine collineation into a path satisfying the more restricted equations (1.2). Using the notation of § 5, it follows from (5.7) that

$$(8.1)\qquad a^i{}_{jk}dx^j dx^k = 0.$$

From (5.10) it is evident that $a^i{}_{jk.l} = a^i{}_{jl.k}$ and since $a^i{}_{jk}$ are homogeneous of degree zero in dx, they satisfy all the conditions of the lemma of § 1. Hence if the vector ξ^i is to define an infinitesimal affine collineation it is necessary and sufficient that it satisfy the equations

$$\xi^i{}_{,j,k} + \xi^l K^i{}_{jkl} + \xi^l{}_{,m} \Gamma^i{}_{jkl} dx^m = 0. \tag{8.2}$$

By a process very analogous to that used in § 5 we find the conditions of integrability of (8.2) to be

$$\xi^h K^i{}_{jkl,h} + \xi^h{}_{,m} dx^m K^i{}_{jkl.h} - \xi^i{}_{,h} K^h{}_{jkl} + \xi^h{}_{,j} K^i{}_{hkl} + \xi^h{}_{,k} K^i{}_{jhl} + \xi^h{}_{,l} K^i{}_{jkh} = 0 \tag{8.3}$$

and

$$\begin{aligned} \xi^h \Gamma^i{}_{jkl,h} + \xi^h{}_{,m} dx^m \Gamma^i{}_{jkl.h} - \xi^i{}_{,h} \Gamma^h{}_{jkl} \\ + \xi^h{}_{,j} \Gamma^i{}_{hkl} + \xi^h{}_{,k} \Gamma^i{}_{jhl} + \xi^h{}_{,l} \Gamma^i{}_{jkh} = 0. \end{aligned} \tag{8.4}$$

By using the coordinate system employed to obtain (5.13) we see that if $\xi^i = a\delta^i{}_1$ is to satisfy (8.2) we must have $(\partial \Gamma^i{}_{jk}/\partial x^1) = 0$. Hence

The most general affine connection admitting a G_1 group of affine collineations may be obtained by taking $H^i(x, \dot{x})$ to be functions of $n - 1$ of *the coordinates x homogeneous of the second degree in $\dot{x}^1, \cdots, \dot{x}^n$.*

If equations (8.2) are to be completely integrable, equations (8.3) and (8.4) must be identities in the $n^2 + n$ quantities $\xi^i{}_{,j}$ and ξ^i. Therefore,

The greatest number of linearly independent affine collineations that a space of paths may admit is $n^2 + n$.

In this case we have from (8.3) and (8.4)

$$K^i{}_{jkl.h} dx^m - \delta^i{}_h K^m{}_{jkl} + \delta^m{}_j K^i{}_{hkl} + \delta^m{}_k K^i{}_{jhl} + \delta^m{}_l K^i{}_{jkh} = 0 \tag{8.5}$$

and

$$\Gamma^i{}_{jkl.h} dx^m - \delta^i{}_h \Gamma^m{}_{jkl} + \delta^m{}_j \Gamma^i{}_{hkl} + \delta^m{}_k{}^i{}_{jhl} + \delta^m{}_l \Gamma^i{}_{jkh} = 0. \tag{8.6}$$

Contracting each set of the above equations for m and h we find

$$K^i{}_{jkl} = 0, \qquad \Gamma^i{}_{jkl} = 0. \tag{8.7}$$

From the second set of (8.7) we see that the affine connection is a function of position (Cf. § 1) and the vanishing of the curvature tensor then shows that the space of paths is flat. Hence

A necessary and sufficient condition that a space of paths must satisfy in order to admit $n^2 + n$ affine collineations is that it be flat.

In a flat space there exists a class of coordinate systems in which the

components of affine connection vanish. In one of these coordinate systems (8.2) become $(\partial^2\xi^i/\partial x^j\partial x^k) = 0$ and therefore $\xi^i = a^i{}_h x^h + b^i$, the finite collineations being $\bar{x}^i = A^i{}_h x^h + B^h$.

If the space is not flat, the existence, and the number, of linearly independent affine collineations can be determined by differentiating (8.3) and (8.4) covariantly and partially with respect to dx and eliminating $\xi^i{}_{,j,k}$ by means of (8.2). We are thus led to a sequence of sets of equations $F^{(1)}, F^{(2)}, \cdots$ which must be compatible for the determination of the quantities ξ^i and $\xi^i{}_{,j}$.

Whether equations (8.2) admit one or more solutions, there exist a number of linear relations between the equations of the sets F so that the number of these equations can be greatly reduced by the following method: let the left hand sides of (8.3) and (8.4) be denoted by $T^i{}_{jkl}$ and $V^i{}_{jkl}$ respectively. Let $T^i{}_{jkl.m,p\ldots}$ and $V^i{}_{jkl,p.m\ldots}$ denote the results of performing the indicated operations and eliminating $\xi^i{}_{,j,k}$ by means of (8.2). Then, by means of (2.9) $V^i{}_{jkl.m}$ has the value

$$V^i{}_{jkl.m} = \xi^h\Gamma^i{}_{jkl.m,h} + \xi^h{}_{,p}\Gamma^i{}_{jkl.m.h}dx^p - \xi^i{}_{,h}\Gamma^h{}_{jkl.m} + \xi^h{}_{,j}\Gamma^i{}_{hkl.m} + \xi^h{}_{,k}\Gamma^i{}_{jhl.m} + \xi^h{}_{,l}\Gamma^i{}_{jkh.m} + \xi^h{}_{,m}\Gamma^i{}_{jkl.h}. \tag{8.8}$$

Evidently $V^i{}_{jkl.m}$ are a set of functions homogeneous of degree -2 in dx and symmetric in all subscripts. We assume

$$V^i{}_{(P)} = \xi^h\Gamma^i{}_{(P),h} + \xi^h{}_{,j}\Gamma^i{}_{(P).h}dx^j - \xi^i{}_{,h}\Gamma^h{}_{(P)} + \sum_{a=1}^{p} \xi^h{}_{,j_a}\Gamma^i{}_{j_1\ldots j_{a-1}hj_{a+1}\ldots j_P} \tag{8.9}$$

where (P) represents the subscripts $j_1, \cdots, j_P$. Partial differentiation of (8.9) with respect to dx and the use of (2.9) and (5.17) in covariant form shows that $V^i{}_{(P).j_{P+1}} = V^i{}_{(P+1)}$; hence reasoning by induction we see that (8.9) are true for any number of subscripts. Moreover $V^i{}_{(P)}$ are homogeneous of degree $-(P-2)$ in dx. Hence if we multiply (8.9) by dx^k and sum for k and any one of the subscripts, say j_P, we obtain

$$V^i{}_{(P)}dx^{j_P} = V^i{}_{(P-1).j_P}dx^{j_P} = -(P-3)V^i{}_{(P-1)}.$$

Therefore all the equations that can be obtained from (8.4) by partial differentiation with respect to dx 0, 1, 2, $\cdots$, r times, are equivalent to the last set. That is, out of $n\left[\binom{n+r+3}{n} - \binom{n+2}{2}\right]$ there are at most $n\left[\binom{n+r+2}{n+1} - \binom{n+1}{1}\right]$ independent equations in these sets.

Differentiating (8.4) covariantly, eliminating second covariant derivatives of ξ^i and using (2.8) and (2.9) we find

$$V^i{}_{jkl,m} = \xi^h \Gamma^i{}_{jkl,m,h} + \xi^h{}_{,r}\Gamma^i{}_{jkl,m.h}dx^r - \xi^i{}_{,h}\Gamma^h{}_{jkl,m}$$
$$+ \xi^h{}_{,j}\Gamma^i{}_{hkl,m} + \xi^h{}_{,k}\Gamma^i{}_{jhl,m} + \xi^h{}_{,l}\Gamma^i{}_{jkh,m} + \xi^h{}_{,m}\Gamma^i{}_{jkl,h}$$

and by a simple computation it can be shown that

$$\begin{aligned}(8.10) \quad V^i{}_{jkl,m.p} - V^i{}_{jkl.p,m} &= V^h{}_{jkl}\Gamma^i{}_{hmp} - V^i{}_{hkl}\Gamma^h{}_{jmp} - V^i{}_{jhl}\Gamma^h{}_{kmp} \\ &- V^i{}_{jkh}\Gamma^h{}_{lmp} + V^i{}_{hmp}\Gamma^h{}_{jkl} - V^h{}_{jmp}\Gamma^i{}_{hkl} - V^h{}_{kmp}\Gamma^i{}_{jhl} - V^h{}_{lmp}\Gamma^i{}_{jkh}.\end{aligned}$$

Therefore the only equations we need consider arising out of (8.4) are those obtainable by covariant differentiation out of $V^i{}_{(r+2)}$, r being the greatest number of partial differentiations with respect to dx.

By means of (2.13) it can be shown that

$$(8.11) \qquad T^i{}_{jkl.m} = V^i{}_{jmk,l} - V^i{}_{jml,k}.$$

Therefore partial differentiation with respect to dx of equations (8.3) does not lead to equations which are linearly independent of those considered above. It can be shown also that

$$\begin{aligned}(8.12) \qquad T^i{}_{jkl,m.p} &= T^i{}_{jkl.p,m} + T^h{}_{jkl}\Gamma^i{}_{hmp} \\ &- T^i{}_{hkl}\Gamma^h{}_{jmp} - T^i{}_{jhl}\Gamma^h{}_{kmp} - T^i{}_{jkh}\Gamma^h{}_{lmp} + V^i{}_{hmp}K^h{}_{jkl} \\ &- V^h{}_{jmp}K^i{}_{hkl} - V^h{}_{kmp}K^i{}_{jhl} - V^h{}_{lmp}K^i{}_{jkh}.\end{aligned}$$

Hence the only equations arising out of (8.3) that we need consider, are those obtained by repeated covariant differentiation. In view of these facts we can state the following theorem:

A necessary and sufficient condition that an affine space of paths must satisfy in order to admit $r(0 < r \leq n^2 + n)$ linearly independent affine collineations is that there exist two integers $N(>0)$ and $P(>0)$ such that the sets of equations $F^{(1)}, \cdots, F^{(N)}$ arising out of $T^i{}_{jkl} = 0$* and *$V^i{}_{(P+2)} = 0$ and the sets $F^{(1)}, \cdots, F^{(N+1)}$ arising out of $T^i{}_{jkl} = 0$ and $V^i{}_{(P+3)} = 0$, shall have matrices of rank $n^2 + n - r$.*

If $\xi^i{}_{(\alpha)}$ and $\xi^i{}_{(\beta)}$ define two linearly independent infinitesimal collineations in an affine space, the components of their alternant are (cf. § 7)

$$(8.13) \qquad \xi^i = \xi^m{}_{(\alpha)}\xi^i{}_{(\beta),m} - \xi^m{}_{(\beta)}\xi^i{}_{(\alpha),m}.$$

By means of (8.2) we obtain

$$\begin{aligned}(8.14) \qquad \xi^i{}_{,j} &= \xi^m{}_{(\alpha),j}\xi^i{}_{(\beta),m} - \xi^m{}_{(\beta),j}\xi^i{}_{(\alpha),m} + \xi^l{}_{(\alpha)}\xi^m{}_{(\beta)}K^i{}_{jml} \\ &+ \Gamma^i{}_{jlm}(\xi^m{}_{(\beta)}\xi^l{}_{(\alpha),h} - \xi^m{}_{(\alpha)}\xi^l{}_{(\beta),h})\,dx^h\end{aligned}$$

---

* By "arising" we mean "obtained by covariant differentiation *alone* and elimination of $\xi^i{}_{,j,k}$ by means of (8.2)."

and

$$
\begin{aligned}
(8.15) \qquad \xi^i{}_{,j,k} = {} & \xi^i{}_{(\alpha),m}(K^m{}_{jkh}\xi^h{}_{(\beta)} + \xi^h{}_{(\beta),l}dx^l\Gamma^m{}_{jkh}) \\
& - \xi^i{}_{(\beta),m}(K^m{}_{jkh}\xi^h{}_{(\alpha)} + \xi^h{}_{(\alpha),l}dx^l\Gamma^m{}_{jkh}) \\
& + \xi^m{}_{(\beta),j}(\xi^h{}_{(\alpha)}K^i{}_{mkh} + \xi^h{}_{(\alpha),l}dx^l\Gamma^i{}_{mkh}) \\
& - \xi^m{}_{(\alpha),j}(\xi^h{}_{(\beta)}K^i{}_{mkh} + \xi^h{}_{(\beta),l}dx^l\Gamma^i{}_{mkh}) \\
& + \xi^l{}_{(\alpha),k}\xi^m{}_{(\beta)}K^i{}_{jml} - \xi^l{}_{(\alpha)}\xi^m{}_{(\beta),k}K^i{}_{jml} \\
& + dx^h\Gamma^i{}_{jml.k}(\xi^m{}_{(\beta)}\xi^l{}_{(\alpha),h} - \xi^m{}_{(\alpha)}\xi^l{}_{(\beta),h}) \\
& + dx^h\Gamma^i{}_{jml}(\xi^m{}_{(\beta),k}\xi^l{}_{(\alpha),h} - \xi^m{}_{(\alpha),k}\xi^l{}_{(\beta),h}) \\
& + (\xi^m{}_{(\alpha)}\xi^p{}_{(\beta)} - \xi^m{}_{(\beta)}\xi^p{}_{(\alpha)})K^l{}_{hkp}\Gamma^i{}_{jml}dx^h.
\end{aligned}
$$

When to (8.15) we add the expressions for $\xi^l K^i{}_{jkl}$ and $\xi^l{}_{,h}dx^h\Gamma^i{}_{jkl}$ we obtain

$$
\begin{aligned}
\xi^i{}_{,j,k} + \xi^l K^i{}_{jkl} + \xi^l{}_{,h}dx^h\Gamma^i{}_{jkl} = {} & \xi^h{}_{(\alpha)}T^i{}_{(\beta)jkh} - \xi^h{}_{(\beta)}T^i{}_{(\alpha)jkh} \\
& + (\xi^h{}_{(\alpha),l}V^i{}_{(\beta)jkh} - \xi^h{}_{(\beta),l}V^i{}_{(\alpha)jkh})dx^l,
\end{aligned}
$$

the remaining terms vanishing because of (2.12) and (2.13). Now $\xi^i{}_{(\alpha)}$ and $\xi^i{}_{(\beta)}$ are assumed to satisfy (8.2) and therefore also (8.3) and (8.4). It therefore follows that ξ^i satisfies (8.2) and we have

An affine space admitting r linearly independent infinitesimal collineations admits the finite continuous group G_r of collineations.

9. *Collineations in Spaces of Paths whose Connection is a Function of Position.* If $\Gamma^i{}_{jkl} = 0$ all functions used in the preceding paragraphs, which are homogeneous of degree zero become free of dx while those which are homogeneous of negative degree vanish. The equations for affine collineations are then

$$(9.1) \qquad \xi^i{}_{,j,k} + \xi^l B^i{}_{jkl} = 0$$

and for projective collineations *

$$(9.2) \qquad \xi^i{}_{,j,k} + \xi^l B^i{}_{jkl} = \delta^i{}_j\phi_k + \delta^i{}_k\phi_j,$$

a point of difference between the two being in the fact that in the first case we are concerned with partial differential equations which are homogeneous in ξ^i while in the second case they are not. It is however possible to reduce the problem of projective collineations to the affine case in one of two ways; either by the use of the generalized projective connection of Veblen or by means of the associated space of T. Y. Thomas.†

We shall make use of the first mentioned method. Briefly stated the

* Cf. M. S. Knebelman, *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 396-400.

† Cf. O. Veblen, *Proceedings of the National Academy of Sciences*, Vol. 14 (1928) pp. 154-166; T. Y. Thomas, *Mathematische Zeitschrift*, Vol. 25 (1926), pp. 723-733.

generalized projective connection of Veblen is an invariant whose components $\Pi^\alpha{}_{\beta\gamma}$ transform according to the law

$$(9.3) \qquad \bar{\Pi}^\alpha{}_{\beta\gamma} = \Pi^\rho{}_{\lambda\mu} v^\alpha{}_\rho u^\lambda{}_\beta u^\mu{}_\gamma + v^\alpha{}_\rho (\partial u^\rho{}_\beta / \partial \bar{x}^i) \delta^i{}_\gamma,$$

where Greek indices are on the range $0, 1, \cdots, n$ while Latin ones are on the range $1, \cdots, n$ and

$$(9.4) \qquad u_0{}^\alpha = \delta_0{}^\alpha, \qquad u_j{}^i = (\partial x^i / \partial \bar{x}^j), \qquad u_j{}^0 = -[\partial \log u^{1/(n+1)} / \partial \bar{x}^j],$$

u being the jacobian of the transformation and $v^\alpha{}_\beta$ is the normalized cofactor of $u^\alpha{}_\beta$, i. e., $u^\alpha{}_\beta v^\beta{}_\gamma = \delta^\alpha{}_\gamma$. This projective connection is more general than the one previously used as no conditions on the Π's need be imposed; but in order to suit it to a geometry of paths the following conditions are imposed on the Π's:

$$(9.5) \qquad \begin{gathered} \Pi^\alpha{}_{\beta\gamma} = \Pi^\alpha{}_{\gamma\beta}, \qquad \Pi^i{}_{ij} = 0, \qquad \Pi^\alpha{}_{\beta 0} = \delta^\alpha{}_\beta, \\ \Pi^0{}_{jk} = -[1/(n-1)]\,[(\partial \Pi^i{}_{jk} / \partial x^i) - \Pi^i{}_{lj} \Pi^l{}_{ik}]. \end{gathered}$$

The components $\Pi^i{}_{jk}$ of the generalized connection transform according to the law (4.3), $\Pi^0{}_{jk}$ being $-r_{jk}$ of § 4 for connections of position.

We may look upon a projective collineation as a transformation which upon carrying a point x to the point $\bar{x}$ carries $\Pi^i{}_{jk}(x)$ into $\Pi^i{}_{jk}(\bar{x})$. We must therefore consider the existence of solutions of equations (9.3)—with $\bar{\Pi}^\alpha{}_{\beta\gamma}(\bar{x})$ replaced by $\Pi^\alpha{}_{\beta\gamma}(\bar{x})$—regarded as second order partial differential equations in the n dependent variables x and the n independent variables $\bar{x}$. As previously we consider the infinitesimal transformation

$$(9.6) \qquad x^i = \bar{x}^i + \xi^i(\bar{x})\,\delta t$$

from which we obtain

$$(9.7) \qquad (\partial x^i / \partial \bar{x}^j) = \delta_j{}^i + (\partial \xi^i / \partial \bar{x}^j)\,\delta t$$

and this gives *

$$(9.8) \qquad (\partial u / \partial \bar{x}^j) = (\partial^2 \xi^h / \partial \bar{x}^h \partial \bar{x}^j)\,\delta t.$$

The vector ξ^i in (9.6) as an affine vector †; to make it projective we introduce the component ξ^0 defined by

$$(9.9) \qquad \xi^0 = -\frac{1}{n+1}\,\frac{\partial \xi^h}{\partial \bar{x}^h}$$

so that (9.4) become, for our infinitesimal transformation

$$(9.10) \qquad u_\beta{}^\alpha = \delta_\beta{}^\alpha + (\partial \xi^\alpha / \partial \bar{x}^j) \cdot \delta_\beta{}^j \cdot \delta t.$$

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* Cf. Eisenhart and Knebelman, *loc. cit.*, p. 41.

† Cf. O. Veblen, *loc. cit.*

By Taylor's expansion—we assume of course that the Π's are analytic functions of the x's—we have

$$\Pi^\alpha{}_{\beta\gamma}(x) = \Pi^\alpha{}_{\beta\gamma}(\bar{x}) + [\partial\Pi^\alpha{}_{\beta\gamma}(\bar{x})/\partial\bar{x}^l]\xi^l\delta t + \cdots.$$

Substitution of these results into (9.3) gives (we drop the bars from over the x's and Π's)

$$(9.11)\qquad \begin{aligned}\delta^j{}_\beta\delta^k{}_\gamma(\partial^2\xi^\alpha/\partial x^j\partial x^k) + \Pi^\alpha{}_{\lambda\gamma}(\partial\xi^\lambda/\partial x^j)\delta_\beta{}^j + \Pi^\alpha{}_{\beta\lambda}(\partial\xi^\lambda/\partial x^j)\delta_\gamma{}^j \\ -\Pi^j{}_{\beta\gamma}(\partial\xi^\alpha/\partial x^j) + \xi^\lambda(\partial\Pi^\alpha{}_{\beta\gamma}/\partial\bar{x}^l) = 0.\end{aligned}$$

If β or γ or both are zero (9.11) vanish identically and if $\alpha = 0$ we get (5.21) (the term $r_{jkh} \equiv 0$ for connections of position). When equations (9.11) are combined with the second *projective derivative* * of ξ^α—whose components are given by

$$\begin{aligned}\xi^\alpha{}_{,\beta,\gamma} = (\partial^2\xi^\alpha/\partial x^j\partial x^k)\delta^j{}_\beta\delta^k{}_\gamma + \Pi^\alpha{}_{\lambda\beta}(\partial\xi^\lambda/\partial x^k)\delta^k{}_\gamma + \xi^\lambda(\partial\Pi^\alpha{}_{\lambda\beta}/\partial x^k)\delta^k{}_\gamma \\ + \Pi^\alpha{}_{\lambda\gamma}[(\partial\xi^\lambda/\partial x^j)\delta^j{}_\beta + \xi^\mu\Pi^\lambda{}_{\mu\beta}] - \Pi^\lambda{}_{\beta\gamma}[(\partial\xi^\alpha/\partial x^j)\delta^j{}_\lambda + \xi^\mu\Pi^\alpha{}_{\mu\lambda}]\end{aligned}$$

we get

$$(9.12)\qquad \xi^\alpha{}_{,\beta,\gamma} + \xi^\mu B^\alpha{}_{\beta\gamma\mu} = 0,$$

$B^\alpha{}_{\beta\gamma\mu}$ being the components of the *projective curvature tensor.* Equations (9.12) are formally identical with (9.1) and the process of projective differentiation being formally identical with covariant differentiation as it is ordinarily defined, we have reduced the problem of projective collineations to one of affine collineations.

It may be of some interest to count the possible number of solutions of (9.12). Suppose the equations of the sets $F^{(1)}, \cdots, F^{(N)}$ of the last theorem but one of § 8 are of rank r; according to that theorem (9.12) will admit $(n+1)^2 + (n+1) - r$ linearly independent solutions. But in this case $(n+1)+1$ linearly independent conditions are imposed on each solution, namely the equations giving the values of $\xi^0{}_{,\alpha}$ and the one equation (9.9). Hence the number of solutions is $n^2 + 2n - r$ which is in agreement with the result obtained in § 6.

The basic advantage of this method of treating projective collineations is the fact that the equations used are *projective tensor* equations, which is not the case with most of the equations used in §§ 4-7. The generalization of this projective connection to one of position and direction should not prove difficult.

* The term piojective derivative is here used in Veblen's sense. Cf. *loc. cit.*, §§ 6 and 7.

10. *Generalized Metric Space. Equations of Motion.* In the preceding paragraphs we have been dealing with spaces of paths or affinely connected manifolds. We shall now consider an n-dimensional continuum in which there exists an absolute scalar differential invariant $F(x, dx)$, positively homogeneous of the second degree in dx and subject to the condition $| F_{.i.j} | \neq 0$. The invariant $F(x, dx)$ is called a *metric* and a space in which it exists *a generalized metric space.**

An affinely connected space may be metric in more than one way; that is, it may admit more than one scalar invariant of the type of $F(x, dx)$. We are not concerned with that question here; we assume the existence of $F(x, dx)$ and obtain the affine properties of the space in terms of it.

We assume that the coordinates of a point are real and that a curve is given by $x^i = f^i(t)$, $f(t)$ being analytic functions of t not all constants. The element of arc ds is defined by

$$ds^2 = eF(x, \dot{x})\,dt, \tag{10.1}$$

where e is plus or minus one according as F is positive or negative. By Euler's theorem on homogeneous functions we have

$$F(x, \dot{x}) = \tfrac{1}{2}F_{.i.j}\dot{x}^i\dot{x}^j. \tag{10.2}$$

Hence if we let $\frac{1}{2}F_{.i.j} = g_{ij}$ (10.1) becomes

$$\dot{s}^2 = eg_{ij}\dot{x}^i\dot{x}^j, \tag{10.3}$$

where, obviously, g_{ij} are symmetric in the subscripts, homogeneous of degree zero in $\dot{x}$ and since F is a scalar, g_{ij} are the components of a tensor. It also follows easily from the law of transformation of a tensor that $g_{ijk} (\equiv g_{ij.k})$ is a tensor symmetric in its subscripts and homogeneous of degree -1 in $\dot{x}$.

When we minimize $\int_{t_0}^{t_1} \dot{s}dt$ we obtain, as in Riemannian geometry, the equations of the extremals, or geodesics,

$$\frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \tag{10.4}$$

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* The study of homogeneous differential forms other than quadratic polynomials dates back to R. Lipschitz, *Crelle*, Vol. 70 (1869), and Vol. 71, 72 (1870), where covariant differentiation first made its appearance. A complete system of invariants—the normal tensors—was developed by Emmy Noether in the *Göttinger Nachrichten*, Vol. 25 (1918); but the study of the geometry of a generalized metric space begins with P. Finsler's Dissertation, *Göttingen* (1918). It is probably for this reason that some authors call a space with a generalized metric a Finsler space.

where for simplicity's sake the arc s is used for a parameter, $\left\{ {i \atop jk} \right\}$ being the Christoffel symbols of the second kind formed out of $g_{ij}[x, (dx/ds)] \cdot \left\{ {i \atop jk} \right\}$ being homogeneous of degree zero in $\dot{x}, \left\{ {i \atop jk} \right\} \dot{x}^j \dot{x}^k$ are the functions H^i of § 1; from this fact we get by partial differentiation *

$$(10.5) \qquad \Gamma^i{}_{jk} = \left\{ {i \atop jk} \right\} + \dot{x}^a \left\{ {i \atop ak} \right\}_{.j} + \dot{x}^\beta \left\{ {i \atop j\beta} \right\}_{.k} + \tfrac{1}{2} \dot{x}^a \dot{x}^\beta \left\{ {i \atop a\beta} \right\}_{.j.k}$$

as the components of the fundamental affine connection of the metric space. From the definition of covariant differentiation of § 2, it follows that †

$$(10.6) \qquad g_{ij,k} = g_{a\beta} \dot{x}^\beta \Gamma^a{}_{ijk}$$

where it is to be understood that $\Gamma^a{}_{ij}$ are the components of the fundamental affine connection as defined by (10. 5).

By a *motion* in a metric space we understand a member of a continuous group of point transformations which preserves the arc-length of every curve. We consider, as in the case of collineations, the effect of an infinitesimal transformation defined by a vector ξ^i on (10. 3). From the definition of a geodesic it follows that if the point transformation is to be a motion it must carry geodesics into geodesics, and since the arc-length of every curve is preserved, the vector ξ^i must satisfy equations (8. 2). Equations (8. 2) however do not constitute a set of necessary and sufficient conditions that the vector ξ^i must satisfy in order to define an infinitesimal motion; to obtain these we have:

$$\begin{aligned} \dot{\bar{s}}^2 &= e\bar{g}_{ij}(\bar{x}, \dot{\bar{x}}) \dot{\bar{x}}^i \dot{\bar{x}}^j \\ \bar{x}^i &= x^i + \xi^i(x)\,\delta u \\ \dot{\bar{x}}^i &= \dot{x}^i + (\partial \xi^i / \partial x^j) \dot{x}^j \,\delta u \\ \bar{g}_{ij}(\bar{x}, \dot{\bar{x}}) &= g_{ij} + (\partial g_{ij}/\partial x^k) \xi^k \delta u + g_{ijk} (\partial \xi^k / \partial x^l) \dot{x}^l \delta u + \cdots. \end{aligned}$$

Hence

$$\begin{aligned} \dot{\bar{s}}^2 = e[g_{ij} &+ (\partial g_{ij}/\partial x^k) \xi^k \delta u + g_{ijk} (\partial \xi^k/\partial x^l) \dot{x}^l \,\delta u + \cdots] \\ &[\dot{x}^i + (\partial \xi^i/\partial x^h)\, \dot{x}^h \delta u]\, [\dot{x}^j + (\partial \xi^j / \partial x^m)\, \dot{x}^m \delta u] \end{aligned}$$

from which we get, upon neglecting higher powers of δ_u than the first,

$$\dot{\bar{s}}^2 = \dot{s}^2 + e h_{ij} \dot{x}^i \dot{x}^j \delta u$$

* Cf. Berwald, *Jahresbericht der Deutschen Mathematiker Vereinigung*, Vol. 34 (1925), pp. 213-20.

† Cf. Berwald, *Mathematische Zeitschrift*, Vol. 25 (1926), p. 54.

where

(10. 7) $h_{ij} = g_{ih}(\partial\xi^h/\partial x^j) + g_{hj}(\partial\xi^h/\partial x^i) + \xi^h(\partial g_{ij}/\partial x^h) + g_{ijh}(\partial\xi^h/\partial x^l)\dot{x}^l.$

The vanishing of $h_{ij}\dot{x}^i\dot{x}^j$ is a necessary and sufficient condition that ξ^i must satisfy in order to define an infinitesimal motion. It is not difficult to verify the fact that the functions h_{ij} satisfy all conditions of the lemma of § 1. Hence we have

If a vector $\xi^i(x)$ is to define an infinitesimal motion, it must satisfy the $n(n+1)/2$ equations

$$g_{ih}(\partial\xi^h/\partial x^j) + g_{hj}(\partial\xi^h/\partial x^i) + \xi^h(\partial g_{ij}/\partial x^h) + g_{ijh}(\partial\xi^h/\partial x^l)\,\dot{x}^l = 0.$$

When in the above equations the partial derivatives are replaced in terms of covariant ones, these equations become

$$g_{ih}\xi^h{}_{,j} + g_{hj}\xi^h{}_{,i} + \xi^h g_{ij,h} + \xi^h{}_{,m}\dot{x}^m g_{ijh} = 0. \tag{10. 8}$$

We shall refer to (10. 8) as the equations of Killing who obtained them for a Riemann space.* From (10. 8) it is evident that h_{ij} are the components of a tensor. We make use of this fact in obtaining metric spaces which admit a given infinitesimal motion. For the given vector $\xi^i(x)$ defining the motion can be normalized so that its components are, say, $\delta_1{}^i$. Then (10. 8) become $\partial g_{ij}/\partial x^1 = 0$. Hence we have

The most general metric space admitting an infinitesimal motion may be obtained by choosing for $F(x, \dot{x})$ a function of $x^2, \cdots, x^n$ *homogeneous of the second degree in $\dot{x}^1, \cdots, \dot{x}^n$, such that* $F_{.1} \neq 0$.

Since the finite transformations generated by the infinitesimal one defined by $\delta_1{}^i$ are $\bar{x}^i = x^i + \delta_1{}^i a$ and g_{ij} is free of x^1, it follows that the arc is preserved under the finite transformations. Hence

A space admitting an infinitesimal motion admits the one parameter finite continuous group G_1 of motions generated by the infinitesimal one.

Two infinitesimal motions are said to be linearly independent if there exists no relation of the form $a\xi^i + b\eta^i = 0$ a and b being constants $(a, b \neq 0)$. By the *path-curves* of a motion we understand the curves of the congruence determined by the vector defining the motion. We then inquire whether two linearly independent motions may have the same path curves. To answer this

* W. Killing, *Crelle's Journal*, Vol. 109, pp. 121-186.

question we normalize the vector defining one motion so that its components are $\delta_1{}^i$; then if the second motion is to have the same path-curves its components must be $\phi(x)\delta_1{}^i$, which when substituted into Killing's equations give

$$(10.9) \qquad g_{i1}(\partial\phi/\partial x^j) + g_{1j}(\partial\phi/\partial x^i) + g_{ij1}(\partial\phi/\partial x^l)\dot{x}^l = 0.$$

When these are multiplied by $\dot{x}^i\dot{x}^j$ and summed for i and for j we get $F_{.1}(d\phi/dt) = 0$ and since $F_{.1} \neq 0$ (cf. last theorem but one above) we must have $\phi = \text{const.}$ which makes the motions linearly dependent. Hence

Two linearly independent motions can not have the same path-curves.

In a general metric space the length of a vector and the angle between two vectors are defined with respect to a direction.* The length of a vector ξ^i relative to a direction dx is given by

$$(10.10) \qquad l = \sqrt{e g_{ij}(x, dx)\xi^i\xi^j}$$

while the angle between two vectors ξ^i and η^i relative to a direction dx is given by

$$(10.11) \qquad \cos\theta = \frac{g_{ij}(x, dx)\xi^i\eta^j}{\sqrt{e_1 g_{ij}\xi^i\xi^j}\ \sqrt{e_2 g_{kl}\eta^k\eta^l}}$$

where it is understood that neither ξ^i nor η^i is a null vector relative to the direction dx.

A motion will be said to be *translatory* or a translation if its path-curves are geodesics † and our next problem is to determine a method for obtaining a general metric space admitting a one parameter group of translations. We first consider the necessary and sufficient conditions that the metric F must satisfy in order that the curves of parameter x^1 (i. e. the curves $x^2 = c^2$, $x^3 = c^3, \cdots, x^n = c^n$) shall be geodesics. Our coordinate

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* Cf. L. Berwald, *Mathematische Zeitschrift*, Vol. 25 (1926), p. 56; J. L. Synge, *Transactions of the American Mathematical Society*, Vol. 29 (1925), pp. 61-68.

† A translation in Riemann spaces is defined as a motion in which each point moves through the same distance and our definition is proved as a theorem. It is just as easy to prove that if the path-curves of a motion are geodesics each point moves through the same distance, the main reason for the adopted definition being the fact that it applies to transatory collineations where no distance is involved. Cf. *Riemannian Geometry*, § 72.

system may be so chosen that x^1 is the arc of the curves of parameter x^1 * and if these curves are to be geodesics we must have

$$\frac{d^2x^i}{dx^{12}} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{dx^1} \frac{dx^k}{dx^1} = 0,$$

which reduce to $\left\{ \begin{matrix} i \\ 11 \end{matrix} \right\} = 0$. From the last equation we have upon multiplying by g_{ih} and summing for i, $[11, h] = 0$ where $[ij, h]$ are the Christoffel symbols of the first kind. Hence our condition becomes

$$\partial g_{1h}/\partial x^1 - \tfrac{1}{2}\partial g_{11}/\partial x^h = 0. \tag{10.12}$$

Since x^1 is to be the arc of these curves we have from (10.3) $g_{11} = e$. This and (10.12) gives $(\partial g_{1h}/\partial x^1) = 0$ that is, g_{1h} is free of x^1. From these results we see that $F(= g_{ij}\dot{x}^i\dot{x}^j)$ must be of the form $a(\dot{x}^1)^2 + \dot{x}^1\phi_1 + \phi_2$ where ϕ_1 is homogeneous of the first degree in $\dot{x}^2, \cdots, \dot{x}^n$ and is free of x^1 while ϕ_2 is homogeneous of degree two in $\dot{x}^2, \cdots, \dot{x}^n$. That this condition on F is sufficient may be proved very easily by showing that the curves $x^2 = c^2$, $x^3 = c^3, \cdots, x^n = c^n$ are geodesics of this metric space.

Combining this result with the second theorem of this section, we have

A space admitting a G_1 finite continuous group of translations may be obtained by taking F in the form $a(\dot{x}^1)^2 + \dot{x}^1\phi_1 + \phi_2$, ϕ_1 and ϕ_2 being functions of $x^2, \cdots, x^n$ and of $\dot{x}^2, \cdots, \dot{x}^n$ homogeneous in the last set of $n-1$ variables of degrees one and two respectively, while a is an arbitrary constant.

We shall next prove a generalization of a theorem on translations in Riemann spaces,† which states that

The angle between the path-curves of a translation and any geodesic (with respect to the geodesic) does not vary along the geodesic.

We take the metric of the space admitting a translation, in accordance with the theorem established above, to be $F = (\dot{x}^1)^2 + \dot{x}^1\phi_1 + \phi_2$ so that

* The truth of this statement may be briefly proved as follows: let $x^1 = t$, $x^2 = c^2, \ldots, x^n = c^n$ be the equations of the curve in terms of a parameter t. The element of arc is then $ds^2 = eg_{11}(dx')^2$ and since g_{11} is homogeneous of degree zero in the one variable dx' it is free of dx'. Hence $s = \int [eg_{11}(x')]^{1/2} dx' = f(x')$. We now apply a transformation of coordinates $\bar{x}^1 = f(x^1)$, $\bar{x}^2 = x^2, \ldots, \bar{x}^n = x^n$. In this coordinate system the equations of our curves become $\bar{x}^1 = s$, $\bar{x}^2 = c^2, \ldots$, $\bar{x}^n = c^n$ where s is the arc.

† Cf. L. P. Eisenhart, *Riemannian Geometry*, § 72.

$g_{11} = 1$, $(\partial g_{ij}/\partial x^1) = 0$. Then the vector defining the translation has components $\delta_1{}^i$ and if $x^i = f^i(s)$ are the equations of a geodesic in terms of the arc, the angle in the theorem is given by

$$\text{(10.13)} \qquad \cos\theta = g_{1j}\dot{x}^j$$

as is obvious from (10.11). Now

$$(d\cos\theta/ds) = \dot{x}^j\,[(\partial g_{1j}/\partial x^k)\dot{x}^k + g_{1jk}\dot{x}^{\cdot k}] + g_{1j}\dot{x}^{\cdot j}$$

and since the curve is a geodesic we have $\dot{x}^{\cdot j} = -\Gamma^j{}_{kl}\dot{x}^k\dot{x}^l$. Hence

$$(d\cos\theta/ds) = \dot{x}^j\dot{x}^k\,[(\partial g_{1j}/\partial x^k) - g_{1h}\Gamma^h{}_{jk} - g_{1jh}\Gamma^h{}_{kl}\dot{x}^l].$$

Because $(\partial g_{ij}/\partial x^1) = 0$,

$$g_{hj}\Gamma^h{}_{1k}\dot{x}^k = [1k, j]\,x^k = \tfrac{1}{2}[(\partial g_{1j}/\partial x_k) - (\partial g_{1k}/\partial x^j)]\,\dot{x}^k \quad \text{(cf. (10.5))}.$$

Hence $g_{hj}\Gamma^h{}_{1k}\dot{x}^j\dot{x}^k = 0$ and $(d\cos\theta/ds) = g_{1j,k}\dot{x}^j\dot{x}^k = 0$ by (10.6) and therefore θ is constant along the geodesic.

If ξ^i is any vector and $x^i = f^i(s)$ are the equations of a curve, we define $\mu^i = \xi^i{}_{,j}\dot{x}^j$ as the *associate* of ξ^i with respect to the curve.*

Let ξ^i be a vector determining a motion; its components satisfy Killing's equations and when these (cf. (10.8)) are multiplied by $\dot{x}^i\dot{x}^j$ and summed for i and for j we find, because of (10.6)

$$\text{(10.14)} \qquad g_{hj}\xi^h{}_{,k}\dot{x}^j\dot{x}^k = 0,$$

from which we have $g_{ij}\mu^i\dot{x}^j = 0$ which implies that μ^i is orthogonal to the curve with respect to the curve. Hence

The associate vector with respect to a curve, of a vector determining a motion, is orthogonal to the curve.

The converse of the above theorem is also true; that is, if in a metric space there exists a vector of position whose associate with respect to any curve, is orthogonal to the curve the vector defines a motion. For (10.14) is a necessary and sufficient condition for orthogonality and from it Killing's equations are easily obtainable by two successive partial differentiations with respect to $\dot{x}^l$ and $\dot{x}^m$.

11. *Integrability Conditions of Killing's Equations.* Because of the symmetry of the tensor g_{ij}, the conditions on the vector ξ^i defining a motion consist of $[n(n+1)/2]$ independent equations for the determination of the

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* Cf. L. P. Eisenhart, *Non-Riemannian Geometry*, § 16. The vector μ^i is also called the *derived* vector.

n^2 unknown functions $(\partial\xi^i/\partial x^j)$. We therefore can not solve Killing's equations for $\xi^i_{,j}$ in terms of g_{ij} and ξ^h. Since a motion carries a geodesic into a geodesic and preserves the arc, ξ^i must also satisfy equations (8.2). Hence the problem of obtaining the components of a motion reduces to the solution of

$$\xi^i_{,j,k} + \xi^l K^i_{jkl} + \xi^l_{,h}\dot{x}^h\Gamma^i_{jkl} = 0 \tag{11.1}$$

subject to the conditions

$$g_{ih}\xi^h_{,j} + g_{hj}\xi^h_{,i} + \xi^h g_{ij,h} + \xi^h_{,m}\dot{x}^m g_{ijh} = 0. \tag{11.2}$$

The dependent variables in these equations are the $n^2 + n$ functions $\xi^i_{,j}$ and ξ^i subject to the $[(n^2+n)/2]$ conditions (11.2). Hence

The greatest number of linearly independent motions that a general metric space may admit is $[n(n+1)/2]$.

The conditions of integrability of (11.1) subject to (11.2) consist of all equations obtainable from (11.1) and (11.2) by partial and covariant differentiation and elimination of $\xi^i_{,j,k}$ by means of (11.1). As in the case of affine collineations we can discard a great many of these equations by proving them linearly dependent on the remaining ones.

Thus the conditions of integrability of (11.1) are (8.3) and (8.4). From (11.2) we get by covariant differentiation and the use of (11.1), (2.8) and (2.9),

$$h_{ij,k} \equiv g_{ij,h}\xi^h_{,k} + g_{ih,k}\xi^h_{,j} + g_{hj,k}\xi^h_{,i} + \xi^h g_{ij,k,h} + \xi^h_{,m}\dot{x}^m g_{ij,k.h} = 0. \tag{11.3}$$

When (8.4) are multiplied by $g_{ip}\dot{x}^p$ and summed for i, the result reduces by means of (10.6) and (10.8) to (11.3). Hence (11.3) are linearly dependent on (8.4). Similarly by partial differentiation we get

$$h_{ij.k} \equiv g_{ijh}\xi^h_{,k} + g_{ihk}\xi^h_{,j} + g_{hjk}\xi^h_{,i} + \xi^h g_{ijk,h} + \xi^h_{,m}\dot{x}^m g_{ijk.h} = 0. \tag{11.4}$$

By means of (11.3) and (11.4) it can be shown that

$$h_{ij,k.l} - h_{ij.l,k} = g_{ip}V^p_{jkl} + g_{pj}V^p_{ikl} + \Gamma^p_{ikl}h_{pj} + \Gamma^p_{jkl}h_{pi},$$

where V^p_{jkl} has the significance of § 8. Therefore, since the equations arising from (11.3) are equivalent to those arising from (8.4), the equations arising from (11.4) by covariant differentiation are also equivalent to those arising from (8.4). Hence we have

A necessary and sufficient condition that a general metric space must satisfy in order to admit r linearly independent motions is that there exist three whole numbers M, N and P $(\geqq 0)$ *such that the rank of the matrix of*

$T, T_1 \cdots T_M$; $V_N, V_{N,1} \cdots V_{N,M}$; $h, h_1 \cdots, h_p$ and that of $T, \cdots T_{M+1}$; $V_{N+1}, \cdots, V_{N+1,M+1}$ $h, \cdots, h_{p+1}$ shall be $[n(n+1)/2] - r$.

We have shown in § 8 that if $\xi^i{}_{(\alpha)}$ $(\alpha = 1, \cdots, r)$ are the components of r linearly independent motions, the components of the alternant of any two of them satisfy (11.1). We therefore need only prove that the components of the alternant satisfy Killing's equations in order to prove that these motions form a G_r finite continuous group of motions. When the values of ξ^h and $\xi^h{}_{,m}$ given by (8.13) and (8.14) are put into (11.2) we find

$$\begin{aligned} H_{ij} \equiv\, & g_{ih}\xi^h{}_{,j} + g_{hj}\xi^h{}_{,i} + \xi^h g_{ij,h} + \xi^h{}_{,m}\dot{x}^m g_{ijh} = g_{ih}(\xi^m{}_{(\alpha),j}\xi^h{}_{(\beta),m} - \xi^m{}_{(\beta),j}\xi^h{}_{(\alpha),m}) \\ & + g_{hj}(\xi^m{}_{(\alpha),i}\xi^h{}_{(\beta),m} - \xi^m{}_{(\beta),i}\xi^h{}_{(\alpha),m}) \\ & + \xi^l{}_{(\alpha)}\xi^m{}_{(\beta)}(g_{ih}K^h{}_{jml} + g_{hj}K^h{}_{iml} + g_{ijh}\dot{x}^p K^h{}_{pml}) \\ & + \dot{x}^p(\xi^m{}_{(\beta)}\xi^l{}_{(\alpha),p} - \xi^m{}_{(\alpha)}\xi^l{}_{(\beta),p})(g_{ih}\Gamma^h{}_{jlm} + g_{hj}\Gamma^h{}_{ilm}) \\ & + \dot{x}^p g_{ijh}(\xi^m{}_{(\alpha),p}\xi^h{}_{(\beta),m} - \xi^m{}_{(\beta),p}\xi^h{}_{(\alpha),m}). \end{aligned}$$

From (2.8) and (2.9) we have

$$g_{ih}K^h{}_{jml} + g_{hj}K^h{}_{iml} + g_{ijh}\dot{x}^p K^h{}_{pml} = g_{ij,l,m} - g_{ij,m,l}$$

and

$$g_{ih}\Gamma^h{}_{jlm} + g_{hj}\Gamma^h{}_{ilm} = g_{ijl,m} - g_{ij,m.l}.$$

Therefore, by adding and subtracting suitable terms,

$$\begin{aligned} H_{ij} = & \xi^l{}_{(\alpha)}h_{(\beta)ij,l} - \xi^l{}_{(\beta)}h_{(\alpha)ij,l} + \xi^l{}_{(\alpha),p}\dot{x}^p h_{(\beta)ij.l} - \xi^l{}_{(\beta),p}\dot{x}^p h_{(\alpha)ij.l} \\ & + \xi^l{}_{(\alpha),i}h_{(\beta)jl} - \xi^l{}_{(\beta),i}h_{(\alpha)jl} + \xi^l{}_{(\alpha),j}h_{(\beta)il} - \xi^l{}_{(\beta),j}h_{(\alpha)il} = 0 \end{aligned}$$

because of (11.2), (11.3) and (11.4). This concludes the proof of the statement made above.

12. *Simply transitive groups as groups of motion.* Let* $\xi_a{}^i$ be n linearly independent vectors of position only. Let these vectors be the components of the infinitesimal transformations of a finite continuous group G_n whose constants of composition are $C^a{}_{\beta\gamma}$. Since $|\xi_a{}^i| \neq 0$ the group is simply transitive. We shall prove that

Any simply transitive group in n variables is the group of motions of an infinity of general metric spaces.†

Let $\eta_i{}^a$ be the set of covariant vectors conjugate ‡ to the set $\xi_a{}^i$, that is,

$$(12.1) \qquad \xi_a{}^i\eta_j{}^a = \delta_j{}^i, \qquad \xi_a{}^i\eta_i{}^\beta = \delta_a{}^\beta.$$

* In this section Greek indices will indicate the vector and Latin ones the component. The summation convention applies to both sets of indices.

† For Riemannian spaces this theorem is due to Bianchi, *loc. cit.*, p. 517.

‡ *Non-Riemannian Geometry*, p. 45.

Killing's equations may be written as

$$(12.2) \qquad \frac{\partial g_{ij}}{\partial x^k} = -\eta_k{}^a \left(g_{ih} \frac{\partial \xi_a{}^h}{\partial x^j} + g_{hj} \frac{\partial \xi_a{}^h}{\partial x^i} + \frac{\partial g_{ij}}{\partial \dot{x}^m} \frac{\partial \xi_a{}^m}{\partial x^h} \dot{x}^h \right).$$

We define an invariant J by means of its components

$$(12.3) \qquad J^h{}_{pq} = -\eta_q{}^a (\partial \xi_a{}^h / \partial x^p) = \xi_a{}^h (\partial \eta_q{}^a / \partial x^p).$$

Because $\xi_a{}^i$ generate a group, we have

$$(12.4) \qquad J^h{}_{pq} - J^h{}_{qp} = -c^\gamma{}_{a\beta} \xi_\gamma{}^h \eta_p{}^a \eta_q{}^\beta$$

and by repeated application of (12.3), (12.4) and the relations between the constants of composition of a group, we obtain

$$(12.5) \qquad (\partial J^h{}_{jk} / \partial x^l) - (\partial J^h{}_{jl} / \partial x^k) = J^h{}_{pk} J^p{}_{jl} - J^h{}_{pl} J^p{}_{jk}.^*$$

From the first of (12. 3) we have

$$\partial \xi_a{}^h / \partial x^k = -\xi_a{}^j J^h{}_{kj}, \qquad \partial^2 \xi_a{}^h / \partial x^k \partial x^l = -\xi_a{}^j (\partial J^h{}_{kj} / \partial x^l) + \xi_a{}^p J^j{}_{lp} J^h{}_{kj}$$

and therefore

$$(12.6) \qquad (\partial J^h{}_{kj} / \partial x^l) - (\partial J^h{}_{lj} / \partial x^k) = J^h{}_{kp} J^p{}_{lj} - J^h{}_{lp} J^p{}_{kj}.$$

Equations (12. 2) may now be put in the form

$$(12.7) \qquad \partial g_{ij} / \partial x^k = g_{ih} J^h{}_{jk} + g_{hj} J^h{}_{ik} + (\partial g_{ij} / \partial \dot{x}^m) \dot{x}^h J^m{}_{hk}.$$

The above system of partial differential equations in the n^2 dependent variables g_{ij} is of a type studied by König,† the conditions of complete integrability of (12. 7) being

$$(12.8) \qquad \frac{\partial f_{ijk}}{\partial x^l} - \frac{\partial f_{ijl}}{\partial x^k} + f_{pql} \frac{\partial f_{ijk}}{\partial g_{pq}} - f_{pqk} \frac{\partial f_{ijl}}{\partial g_{pq}} + \frac{\partial f_{ijk}}{\partial g_{pqr}} \frac{\partial f_{pql}}{\partial \dot{x}^r} - \frac{\partial f_{ijl}}{\partial g_{pqr}} \frac{\partial f_{pqk}}{\partial \dot{x}^r}$$
$$+ \left(\frac{\partial f_{ijk}}{\partial g_{pqs}} \frac{\partial f_{pql}}{\partial g_{tu}} - \frac{\partial f_{ijl}}{\partial g_{pqs}} \frac{\partial f_{pqk}}{\partial g_{tu}} \right) g_{tus} = 0$$

and

$$(12.9) \qquad \frac{\partial f_{ijk}}{\partial g_{pqr}} \frac{\partial f_{pql}}{\partial g_{hms}} - \frac{\partial f_{ijl}}{\partial g_{pqr}} \frac{\partial f_{pqk}}{\partial g_{hms}} + \frac{\partial f_{ijk}}{\partial g_{pqs}} \frac{\partial f_{pql}}{\partial g_{hmr}} - \frac{\partial f_{ijl}}{\partial g_{pqs}} \frac{\partial f_{pqk}}{\partial g_{hmr}} = 0,$$

where f_{ijk} is the right-hand side of (12. 7) and x^i, $\dot{x}^i$, g_{ij} and g_{ijk} are regarded as independent variables in the partial derivatives of (12. 8) and (12. 9). From (12. 7) we have

$$(12.10) \qquad \partial f_{ijk} / \partial g_{pqr} = \delta_i{}^p \delta_j{}^q J^r{}_{tk} \dot{x}^t$$

and when this is put into (12. 9) it is found that these conditions are satisfied identically. Again

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* Cf. ***Riemannian Geometry***, pp. 247-49.

† J. König, ***Mathematische Annalen***, Vol. 23 (1884), pp. 520-27.

$$
(12.11) \qquad \begin{aligned} \partial f_{ijk}/\partial x^l &= g_{ih}(\partial J^h{}_{jk}/\partial x^l) + g_{hj}(\partial J^h{}_{ik}/\partial x^l) + g_{ijm}(\partial J^m{}_{hk}/\partial x^l)\dot{x}^h, \\ \partial f_{pqk}/\partial \dot{x}^r &= g_{pqm}J^m{}_{rk}, \qquad \partial f_{ijk}/\partial g_{pq} = \delta_i{}^p J^q{}_{jk} + \delta_j{}^q J^p{}_{ik}. \end{aligned}
$$

When these values of the partial derivatives of f are put into (12.8) the result is

$$
\begin{aligned} & g_{ih}[(\partial J^h{}_{jk}/\partial x^l) - (\partial J^h{}_{jl}/\partial x^k) + J^h{}_{pl}J^p{}_{jk} - J^h{}_{pk}J^p{}_{jl}] \\ & + g_{hj}[(\partial J^h{}_{ik}/\partial x^l) - (\partial J^h{}_{il}/\partial x^k) + J^h{}_{pl}J^p{}_{ik} - J^h{}_{pk}J^p{}_{il}] \\ & + g_{ijm}[(\partial J^m{}_{hk}/\partial x^l) - (\partial J^m{}_{hl}/\partial x^k) + J^m{}_{pl}J^p{}_{hk} - J^m{}_{pk}J^p{}_{hl})\dot{x}^h = 0 \end{aligned}
$$

which is satisfied identically because of (12.5). Hence (12.2) are completely integrable, that is, they admit a solution which for $x^i = x_0{}^i$ are arbitrary functions of $\dot{x}^i$, which proves the theorem.

From the fact that a vector defining a motion satisfies not only (12.2) but also (8.2) it follows that the above theorem will be true if we replace in it the words " motion " and " metric space " by " collineations " and " space of paths " respectively.

The Complete System of Two Quaternary Quadratics.

By J. Williamson.

1. *Introduction.* The system of concomitants for two quaternary quadratic forms was worked out by Gordan in the *Mathematische Annalen,* Bd. 56. Gordan's list of 580 forms was reduced by Turnbull to 125 forms.* This paper, while only reducing the number from 125 to 122, finds the concomitants by quite a different method. The method shows the fundamental rôle played by the four quadratic covariants in determining the complete system and gives very simply all the formulae of reduction used by Turnbull. Further, it seems likely that without much difficulty the method could be extended to the case of two quadratics in five or six variables.

2. List A.

The Prepared System.

$$1_x = a_x, \qquad 2_x = (A\beta x), \qquad 3_x = (B\alpha x), \qquad 4_x = b_x.$$

$$(12) = a_\beta(Ap), \qquad (23) = (AB)(\alpha\beta p),$$

$$(43) = b_\alpha(Bp), \qquad (14) = (abp),$$

$$(13) = (aB\alpha p) = \dot{b}_\alpha(a\dot{b}'p), \qquad (42) = (bA\beta p).$$

$$(123) = -a_\beta(AB)u_\alpha, \qquad (432) = -b_\alpha(AB)u_\beta,$$

$$(124) = a_\beta(Abu), \qquad (431) = b_\alpha(Bau).$$

$$(1234) = -a_\beta(AB)b_\alpha, \qquad (12, 43) = a_\beta b_\alpha(ABux).$$

List B.

The 122 unreduced forms.

5 Invariants	$a_\alpha^2,\ b_\alpha^2,\ (AB)^2,\ a_\beta^2,\ b_\beta^2.$
5 Covariants	$i_x^2, \qquad (1234)\ 1_x 2_x 3_x 4_x.$
5 Contravariants	$(ijk)^2, \quad (1234)(123)(234)(124)(134).$
16 Complexes	$(ij)^2 \quad 6, \quad (ij)(jk)(ki) \quad 4,$ $(1234)(12)(34), \quad (1234)(14)(23),$ $(1234)(ij)(ik)(im) \qquad 4.$

* H. W. Turnbull, "The Simultaneous System of Two Quadratic Quaternary Forms," *Proceedings of the London Mathematical Society,* Ser. 2, Vol. 18, Parts 1 and 2, pp. 70-94. Further reference to this paper will be indicated simply by "Turnbull's paper."

19 Mixed Forms containing x and u

$(1234)(123)4_x$, $(1234)(234)1_x$,
$(ijk)i_xj_xk_x$ 4
$(1234)(ijk)(ijm)i_xj_x$ 6
$(ijk)(ijm)(ikm)i_x$ 4
$(12, 43)^2$, $(1234)(12, 43)$
$(12, 43)(124)(134)1_x4_x$.

14 Mixed Forms containing x and p

$i_x(ij)j_x$ 6, $(1234)(23)1_x4_x$,
$\left.\begin{array}{l}(1234)(21)(14)1_x3_x\\(1234)(23)(34)1_x3_x\\(1234)(34)1_x2_x\end{array}\right\}$ and three similar forms.
$(1234)(12)(23)(34)2_x3_x$.

14 Mixed Forms containing p and u

$(1234)(ijk)(ijm)(ij)$ 6, $(123)(234)(14)$,
$\left.\begin{array}{l}(134)(342)(12)\\(1234)(123)(134)(12)(23)\\(1234)(123)(134)(14)(43)\end{array}\right\}$ and three similar forms.
$(124)(134)(12)(14)(34)$.

44 Mixed Forms containing x, p and u

$(123)(12)3_x$, $(123)(23)1_x$, $(124)(12)4_x$,
$(124)(14)2_x$, $(1234)(123)(41)1_x$
$(1234)(123)(43)3_x$, $(1234)(124)(32)2_x$,
$(1234)(124)(34)4_x$, $(132)(23)(14)4_x$, and nine similar forms.
$(143)(23)(12)4_x$,
$(ijk)(ij)(ik)i_x$ 12,
$\left.\begin{array}{l}(1234)(124)(14)(34)1_x,\\(1234)(123)(23)(34)2_x,\\(1234)(124)(12)(23)1_x,\\(1234)(234)(42)(21)4_x,\\(1234)(124)(24)(43)2_x,\end{array}\right\}$ and five similar forms.
$(12, 43)(12)(34)$,
$(12, 43)(124)(34)4_x$, and a similar form.

In this list i, j, k, m take the values 1, 2, 3, 4. The number appearing on the right of a concomitant is the number of distinct concomitants of that type. A similar form means a form in which the symbols 1, 4 and 2, 3 are interchanged. The actual complete system is obtained by removing from each form any invariant factor which may appear. For example,

$$(12)^2 = a_\beta^2(Ap)^2.$$

We therefore drop the factor a_β^2 and take $(Ap)^2$ as the unreduced form.

3. *Notation.* Let

$$f = a_x^2 = a'^2_x = \cdots, \qquad g = b_x^2 = b'^2_x = \cdots, \tag{1}$$

be the two quadratics. Further let A, B denote a convolution of two equivalent symbols a and a', b and b' respectively, so that the factor $(aa'cd)$ is written as (Acd). Similarly, let a convolution of three equivalent symbols a, a', a'' be written as α and of three equivalent symbols b, b', b'' as β, so that $(daa'a'')$ is written d_α or $(d\alpha)$.

If x, y, z, t are four sets of cogredient variables, the following notation may be used;

$$\left\{\begin{aligned} &\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} = (ab \mid xy) = (abxy), \\ &\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = (abc \mid xyz) = (abcxyz), \\ &\left| a_x \quad b_y \quad c_z \quad d_t \right| = (abcd \mid xyzt) = (abcd)(xyzt). \end{aligned}\right. \tag{2}$$

Furthermore we have the fundamental identities;

$$\left\{\begin{aligned} (abcd)e_x &= (abce)d_x + (abed)c_x + (aecd)b_x + (ebcd)a_x, \\ (ab \mid xy) &= a_x b_y - a_y b_x, \\ (abc \mid xyz) &= (ab \mid xy)c_z - (ac \mid xy)b_z - (bc \mid xy)a_z, \\ &= (ab \mid xy)c_z - (ab \mid xz)c_y - (ab \mid yz)c_x, \\ (abcd \mid xyzt) &= (abc \mid xyz)d_t - (abd \mid xyz)c_t - (adc \mid xyz)b_t - (dbc \mid xyz)a_t. \end{aligned}\right. \tag{3}$$

These identities may be written more shortly in the forms,

$$\begin{aligned} (\dot{a}\dot{b}\dot{c}\dot{d})\dot{e}_x &= 0, \\ (ab \mid xy) &= \dot{a}_x \dot{b}_y = a_{\dot{x}} b_{\dot{y}}, \\ (abc \mid xyz) &= (\dot{a}\dot{b} \mid xy)\dot{c}_z = (ab \mid \dot{x}\dot{y})c_{\dot{z}}, \\ (abcd \mid xyzt) &= (\dot{a}\dot{b}\dot{c} \mid xyz)\dot{d}_t, \end{aligned}$$

where the dots above the letters indicate the determinantal permutations that must be made.

4. *Co-ordinates.* Let the four vertices of a tetrahedron be the points

$$x, \quad y, \quad z, \quad t \tag{4}$$

and the four planes of the tetrahedron be

$$q, \quad w, \quad v, \quad u \tag{5}$$

where $x = (x_1 x_2 x_3 x_4)$, $q = (q_1 q_2 q_3 q_4)$ with similar expressions for the other points and planes, so that x, y, z, t denote cogredient quaternary symbols each contragredient to any one of q, w, v, u. Let the three planes u, v, w meet at x and so on, as is indicated by the order of writing (4) and (5). Then, if (uvw) denotes the set of four determinants $(u_2 v_3 w_4, u_1 v_4 w_3, u_1 v_2 w_4, u_1 v_3 w_2)$,

$$(6) \qquad \begin{cases} x = (uvw), & \Delta^2 u = (xyz), \\ \Delta^3 = (xyzt), & \Delta(uv)_{12} = -(xy)_{34} \text{ etc.}, \end{cases}$$

where $(qwvu) = \Delta$.

Hence, if d, e, f, g are quaternary symbols cogredient with u,

$$(7) \qquad \begin{cases} \Delta^2(defu) = (def \mid xyz) \\ \Delta(dep) = \Delta(deuv) = -(de \mid xy) \\ \Delta^3(defg) = (defg)(xyzt) = (defg \mid xyzt). \end{cases}$$

5. *Determination of the prepared system.* By the fundamental theorem every concomitant of the two quadratics (1) is a sum of products of factors of four types, d_x, $(deuv)$, $(defu)$ and $(defg)$, where each of the symbols d, e, f, g stands for a symbol a or b and must occur exactly twice in each product. The factor Δ might also occur but as it is a concomitant as it stands, we neglect it.

Hence by (7) every concomitant multiplied by a suitable power of Δ can be represented as a sum of products of factors of the four types d_x, $(de \mid xy)$, $(def \mid xyz)$ and $(defg \mid xyzt)$, where a power of Δ is introduced with each variable y, z, and t.

Let K be such a concomitant multiplied by a suitable power of Δ. If K has the factor $(aa_1a_2a_3 \mid xyzt)$, then by (2) $K = \sum (aa_1a_2a_3 \mid xyzt) a_\xi a_{1\eta} a_{2\zeta} a_{3\delta} M$, where ξ, η, ζ, δ are variables x, y, z, t but not necessarily distinct. By determinantally permuting the equivalent symbols a, a_1, a_2, a_3, we have

$$4!\, K = \sum N$$

where each term N has the factor $(aa_1a_2a_3)^2 (xyzt)^2$ or else is zero. Hence, if K has a factor a_α (or b_β), K is either zero or has the invariant factor a_α^2 (or b_β^2). In either case K may be considered to be reducible, and we denote this by writing $K \equiv 0 \bmod a_\alpha$.

A similar proof shows that, if $a_1 a_2 a_i$ or $b_1 b_2 b_i$ ($i = 2$ or 3) are convolved in K, the complementary symbols may also be convolved or finally:

LEMMA I. *If K is not reducible,*

$$mK = \sum M,$$

where each M is a product of factors of the types a_ξ, b_ξ, $(A \mid \xi\eta)$, $(B \mid \xi\eta)$, $(\alpha \mid \xi\eta\zeta)$, $(\beta \mid \xi\eta\zeta)$ and each symbol a, b, A, B, α, β occurs exactly twice in each product, while m is an integer.

Now two quadratics in n variables have $n+1$ invariants which form a complete system of invariants.* For our case these invariants are $a_\alpha{}^2$, $b_\beta{}^2$, $(AB)^2 = \Theta$, $a_\beta{}^2 = \Phi$, $b_\alpha{}^2 = \Psi$. There are also n quadratic covariants belonging to the complete system of two quadratics in n variables.† Let us denote the four quadratics in our case by,

$$\begin{aligned}&1_x{}^2 = a_x{}^2, \quad 4_x{}^2 = b_x{}^2, \quad \text{(i. e. the two original forms)},\\ &2_x{}^2 = (A\beta x)^2 = (a_\beta a'_x - a'_\beta a_x)^2 \text{ where } A = aa',\\ &3_x{}^2 = (B\alpha x)^2 = (b_\alpha b'_x - b'_\alpha b_x)^2 \text{ where } B = bb'.\end{aligned}$$

Now

$$\begin{aligned}(12 \mid \xi\eta) &= a'_\beta(aa'' \mid \xi\eta) - a''_\beta(aa' \mid \xi\eta) = a_\beta(A \mid \xi\eta) + (aa''a' \mid \xi\eta\beta) \qquad (A = a'a''),\\ &\equiv a_\beta(A \mid \xi\eta) \text{ neglecting the last term.}\end{aligned} \tag{8}$$

Similarly

$$\begin{aligned}(123 \mid \xi\eta\zeta) &\equiv a_\beta(A\dot{a}_1 \mid \xi\eta\zeta)(Ba_2\dot{a}_3) \qquad (\alpha = a_1a_2a_3),\\ &\equiv a_\beta(\alpha \mid \xi\eta\zeta)(AB) \qquad \text{mod } a_\alpha,\end{aligned} \tag{9}$$

and

$$(1234 \mid xyzt) \equiv a_\beta(AB)b_\alpha(xyzt). \tag{10}$$

We have also the corresponding results

$$\begin{cases}(43 \mid \xi\eta) \equiv b_\alpha(B \mid \xi\eta),\\ (432 \mid \xi\eta\zeta) \equiv b_\alpha(BA)(\beta \mid \xi\eta\zeta).\end{cases} \tag{11}$$

Hence, if K be multiplied by a suitable factor I composed of powers of Θ, Φ, and Ψ, it follows by lemma I that mIK can be expressed as a sum of terms, where each term is a product of factors of the one type $i_\xi (i = 1, 2, 3, 4: \xi = x, y, z, t)$. But in K the variables x, y, z, t were originally convolved as x, (xy), (xyz), and $(xyzt)$. Now mIK involves the same variables as K,

---

* H. W. Turnbull, *Theory of Determinants, Matrices and Invariants*, Chap. XX, § 6, page 304.

† Turnbull and Williamson, "The Minimum System of Two Quadratics in n Variables," *Proceedings of the Royal Society of Edinburgh*, Vol. 45 (1924-25), pp. 149-165.

the only difference being that each symbol a or b occurring in K has been replaced by a symbol 1, 2, 3 or 4. Hence we may convolve the variables in mIK back again, exactly as they were originally convolved in K, and therefore have the result that $mIK = \sum M$ where each M is a product of factors of the four types i_x, $(ij \mid xy)$, $(ijk \mid xyz)$ and $(ijkm \mid xyzt) = (ijkm)(xyzt)$ and the symbols A, α, B, β, which occurred in K, also occur in M. We have now proved the theorem:

THEOREM I. *Every concomitant K, is either reducible or else, when multiplied by a suitable invariant factor composed of powers of Θ, Φ and Ψ, it can be expressed as a sum of terms, where each term is a product of factors of the four types mentioned above. Further, each symbol i, j, k, m must occur exactly twice in each product and the symbols A, α, B, β which are convolved in K appear explicitly in the symbols i, j etc.*

If K is reducible $K = I_1 K_1$ where I_1 is an invariant factor composed of powers of a_α^2, b_β^2 and K_1 is not reducible.

Therefore, by considering all possible products M, composed of the four factor types, mentioned in the theorem above, we will obtain a system in terms of which every concomitant K of the two quadratics, if multiplied by an invariant factor, can be expressed. But any such product $M = \sum I_i G_i$, where each G_i is a concomitant involving the variables properly convolved and each I_i is an invariant composed of powers of Θ, Φ, Ψ. Hence, if we determine the complete system of the G_i, we have a system of concomitants in terms of which every concomitant K, if multiplied by a suitable invariant factor, can be expressed.

Let us consider the concomitants G_i in the following order:

1. Consider G_1 before G_2 if G_1 is of less degree in the coefficients of the two quadratics than G_2.

2. If 1 fails to distinguish G_1 and G_2, consider G_1 before G_2 if G_1 has more symbols α (or β) than G_2 and not less symbols β (or α).

3. If both 1 and 2 fail, then consider G_1 before G_2 if G_1 has more symbols A (or B) than G_2 and not less symbols B (or A). That this is a legitimate procedure follows from the fact, that each G_i is also a concomitant K and can therefore always be obtained from symbols i_x etc., in which the symbols α, β, A, B appearing in G_i also appear in the i_x.

It follows immediately from this scheme and the manner in which the formulae (8) to (11) were obtained, that we are at liberty to use these formulae in determining the G_i from the products M. Moreover,

$$\begin{aligned}(23\xi\eta) &= a_\beta(B\dot{a}_2\dot{a}_3)(a'\dot{a}_1 \mid \xi\eta) - a'_\beta(B\dot{a}_2\dot{a}_3)(a\dot{a}_1 \mid \xi\eta) && \alpha = a_1a_2a_3,\\ &\equiv \dot{a}_{3\beta}(B\dot{a}_2 a)(a'\dot{a}_1 \mid \xi\eta) - \dot{a}_{3\beta}(B\dot{a}_2 a')(a\dot{a}_1 \mid \xi\eta) && \text{mod } b_\beta.\\ &\equiv (AB)(\alpha \mid \beta\xi\eta) \qquad \text{mod } a_\alpha.\end{aligned}$$

We next proceed to show that of the factors of M we may neglect those in which equivalent symbols $1\,1'$, $2\,2'$, $3\,3'$, $4\,4'$ occur convolved.

Let us first consider the case in which $1\,1'$ is convolved. Since an invariant factor a_β may be introduced when 12 is convolved in one bracket factor, we have to consider the separate cases:

$$\begin{aligned}&(11' \mid \xi\eta) = (aa' \mid \xi\eta) = (A \mid \xi\eta),\\ &(121' \mid \xi\eta\zeta) \equiv a_\beta(Aa' \mid \xi\eta\zeta) \equiv a_\beta(\alpha \mid \xi\eta\zeta),\\ &(121'2' \mid xyzt) \equiv a_\beta a'_{\beta'}(AA' \mid xyzt) \equiv 0 \text{ mod } a_\alpha,\\ &(1231' \mid xyzt) \equiv a_\beta(AB)(\alpha a' \mid xyzt) \equiv 0 \text{ mod } a_\alpha.\end{aligned}$$

In every case the resulting G_i have been reduced to G's of a simpler type and so the factors above may be neglected. Similarly we do not need to consider factors in which $4\,4'$ appears convolved. The cases to be considered in which $2\,2'$ occurs convolved are

$$22',\quad 122',\quad 232',\quad 2342',\quad 232'3',\quad 1232'.$$

Of these the last three are reducible mod a_α:

$$\begin{aligned}(22' \mid \xi\eta) &= (\dot{a}_1\beta)(a'_1\beta')(\dot{a}_2a'_2 \mid \xi\eta) - (\dot{a}_1\beta)(a'_2\beta')(\dot{a}_2a'_1 \mid \xi\eta),\\ &\qquad\qquad A = a_1a_2, \quad A' = a'_1a'_2,\\ &= (\dot{a}_1\beta)(\dot{a}_2\beta')(A' \mid \xi\eta) - (A'\dot{a}_2 \mid \xi\eta\beta')(\dot{a}_1\beta),\\ &\equiv (A'\dot{a}_2 \mid \beta'\xi\eta)(\dot{a}_1\beta) \text{ mod } b_\beta:\\ (122' \mid \xi\eta\zeta) &\equiv (a\beta)(\dot{a}_1\beta')(A\dot{a}_2 \mid \xi\eta\zeta) \equiv (a\beta)(\dot{a}_1\beta')(\alpha \mid \xi\eta\zeta):\\ (232' \mid \xi\eta\zeta) &\equiv (AB)\,[(\dot{a}\beta)(a'_1\beta')(\dot{a}_1\dot{a}_2a'_2 \mid \xi\eta\zeta) - (\dot{a}\beta)(a'_2\beta')(\dot{a}_1\dot{a}_2a'_1 \mid \xi\eta\zeta)],\\ &\equiv 0 \text{ mod } a_\alpha,\quad b_\beta,\quad \alpha = aa_1a_2,\quad A' = a'_1a'_2.\end{aligned}$$

Once again the resulting G_i have been reduced to G's of a simpler type. Similarly we do not need to consider bracket factors in which $3\,3'$ appears convolved. Furthermore, since

$$i_\xi - i'_\eta = (ii' \mid \xi\eta)$$

we can interchange equivalent symbols i in any product and so do not require to use distinguishing marks on the equivalent symbols i.

But by (7)

$$\Delta^{-1}(ij \mid xy) = (ijuv) = (ijp),$$
$$\Delta^{-2}(ijk \mid xyz) = (ijku),$$
$$\Delta^{-3}(1234 \mid xyzt) = (1234),$$

and, since originally a power of Δ was introduced with each variable y, z, t, we can pass back to the original variables u and uv. If we write (ij) for (ijp) and (ijk) for $(ijku)$ we may state the final result in the theorem;

THEOREM II. *Every concomitant of the two quadratics, if multiplied by a suitable invariant factor, is a sum of terms, where each term is a product of symbolic factors of the four types* (1234), (ijk), (ij), i_x, *together with the invariants* a_α^2 *and* b_β^2 *and each symbol* i, j, k *must occur an even number of times in every product. Moreover no equivalent symbols* i, j, *etc. may occur in the same bracket factor.*

The values of the factors (ij), (ijk) are given in list A § 2. On considering this list we notice that corresponding to convolutions of the symbols 12, 23 and 34 factors a_β, (AB), b_α appear. Since we wish to determine all concomitants (not merely all the concomitants multiplied by an invariant factor), we must see if, in forming the factors (ij), (ijk), we have broken up 12, 23 or 34. This may have happened. Before convolving the variables x, y, z back to their original forms i. e. to xy, xyz, we might have the product $(12 \mid xy)(34 \mid xz)$. In convolving the variables x, y, z together we permute x, y and z to obtain

$$(123 \mid xyz)4_x - (124 \mid xyz)3_x \quad \text{or}$$
$$(324 \mid xyz)1_x - (314 \mid xyz)2_x.$$

In either case we have broken up a factor 12 or 34, and so must introduce the new factor

$$(12, 43) \equiv (123)4_x - (124)3_x,$$
$$\equiv (143)2_x - (243)1_x.$$

This is the only new factor that appears for (ij, kn) is always expressible in terms of the earlier factors, if ij or kn is 13 or 24. Further

$$(12, 23) = -(122)3_x + (123)2_x$$

where (122) has the equivalent symbols 2 2 convolved and so may be neglected and (123) has both 12 and 23 convolved. Similarly no factors of the type (ijk, lm) or (ijk, lmn) need to be introduced. Hence with the introduction of (12, 34) we are now assured that, since we have found all factors in which 12, 23, 34 are convolved, we may, from our list of concomitants multiplied by invariant factors, remove the invariant factors and obtain the actual concomitants.

6. *Identities.* All the identities in Turnbull's paper are obtained from the following:

$$(12)\quad \begin{cases} (ij)k_x = (ik)j_x + (jk)i_x, \\ (12)(34) = (13)(24) + (32)(14), \\ (123)4_x = (124)3_x - (134)2_x - (324)1_x, \\ (ijk)(im) = (ijm)(ik) - (ikm)(ij), \\ (12, 43) = (123)4_x - (124)3_x, \\ \qquad\qquad = (143)2_x - (243)1_x. \end{cases}$$

The first four of these identities are analogous to those for binary and ternary types and the others are a direct result of the definition of (12, 43).
We now deduce two other important identities.

$$(13)\quad \begin{aligned} (14)(12, 43) &= [(123)4_x - (124)3_x]\ (14), \\ &= (124)(13)4_x + (143)(12)4_x - (124)(13)4_x - (124)(34)1_x, \\ &= (143(12)4_x - (124)(34)1_x. \end{aligned}$$

$$(14)\quad \begin{aligned} (23)(12, 43) &= [(123)4_x - (124)3_x]\ (23), \\ &= (123)(43)2_x + (123)(24)3_x - (123)(24)3_x - (324)(21)3_x, \\ &= (123)(43)2_x - (324)(21)3_x. \end{aligned}$$

By means of the above identities all products of factors of the type indicated in theorem II can be expressed in terms of the list given § 2 B. These identities cannot however be applied blindly. Consider for example the identity, $(12)3_x = (13)2_x + (32)1_x$. This identity involves on the left hand side the factor a_β, because (12) is convolved. Accordingly, if 12 is convolved an even number of times in a concomitant, since eventually we shall discard the invariant factors, we are not at liberty to use this identity as it stands.

It may however always be used in the form $(13)2_x = (12)3_x + (23)1_x$ since (13) does not introduce an extraneous factor.

7. *The determination of the concomitants.* It is not our intention here to reproduce the work whereby the list B was obtained. We shall however give a few typical examples.

Complexes. The only factors that may occur are the six (ij) factors and (1234). If the factor (1234) does not occur, the problem of determining the irreducible complexes is strictly analogous to that of binary forms, and accordingly the result is,

The six quadratic complexes $(ij)^2$,
The four cubic complexes $(ij)\ (jk)\ (ki)$.

If the factor (1234) occurs and the form is a quadratic complex there are three possible cases,

$$(1234)(12)(34), \qquad (1234)(23)(14), \qquad (1234)(13)(24),$$

and of these the third is expressible in terms of the other two. If the form is a cubic complex it must be one of the four,

$$(1234)(ij)(ik)(im) \qquad (i, j, k, m, = 1, 2, 3, 4).$$

The only other possibility is

$$(1234)(12)(23)(34)(41)(13)(24)$$

and this is obviously reducible.

Concomitants involving the factor $(12, 43)$. If we consider a concomitant M, involving the factor $(12, 43)$, to be reducible when $(12, 43)$ can be expressed in terms of factors $(ijk)m_x$, any irreducible concomitant involving $(12, 43)$ must have 12 and 34 convolved an even number of times. [Last two of (12)].

Let M denote a concomitant containing $(12, 43)$. Then obviously M may be

$$(12, 43)\ (1234),$$
$$\text{or}\ \ (12, 43)^2.$$

If $M \neq (12, 43)(1234)$, 23 must be convolved an even number of times in M, for otherwise M would have the factor $(AB)(ABux)$. It follows immediately that, if M does not contain the variable p nor the factor (1234), there are only two possibilities for M:

$$M = (12, 43)(123)(234)3_x 2_x,$$
$$\text{or} \qquad = (12, 43)(124)(134)1_x 4_x$$

where in the first 12, 23, 34 are all convolved twice and in the second 12 and 34 are convolved twice.

If $M = (1234)(12, 43)N$, 23 must be convolved once in N, for it cannot be convolved three times. Hence N contains (123) or (234). Therefore

$$N = (123)(124)3_x 4_x \qquad \text{or} \qquad (234)(431)2_x 1_x.$$

But $(123)(124)3_x 4_x = (123)\,[(123)4_x^2 - (12, 43)4_x]$ by (12) with a similar identity in the other case. Hence M is reducible.

If M contains the variable p, M cannot have the factor (14) or (23) (by (13) and (14)). It is now a simple matter to obtain the complete list of irreducible concomitants involving the factor (12, 43). The complete list is,

$(12, 43)^2$
$(12, 43)(1234)$
$(12, 43)(123)(234)3_x 2_x$ $\qquad R$
$(12, 43)(124)(134)1_x 4_x$
$(12, 43)(12)(34)$
$(12, 43)(124)(34)4_x$
$(12, 43)(134)(12)1_x$
$(12, 43)(1234)(231)(12)3_x$ $\qquad R$
$(12, 43)(1234)(234)(34)2_x$ $\qquad R$

8. *Three special reductions.* The concomitants above, which are marked R, appear in Turnbull's list but are reducible. We proceed to show this.

After removing the invariant factor, $(12, 43)(123)(234)3_x 2_x$ is equivalent to

$$(A\beta x)\,(B\alpha x)\,(ABux)u_\alpha u_\beta$$
$$\equiv 2(aa'\beta x)\,(B\alpha x)\,(Bau)u_\alpha u_\beta a'_x$$
$$\equiv 2\,[a_\beta a'_x - a_x a'_\beta]\,(B\alpha x)\,(Bau)u_\alpha u_\beta a'_x.$$

Each term is factorable, the first having the factor a_x^2 and the second $u_\beta a'_\beta a'_x$. Hence the concomitant is reducible.

$$(12, 43)(1234)(231)(12)3_x$$

is equivalent to

$$(ABux)(B\alpha x)(Ap)u_\alpha$$
$$\equiv 2(Abu)(b_\alpha b'_x - b_x b'_\alpha)(Ap)b'_x u_\alpha$$

and again each term has a concomitant factor. An exactly similar proof shows that $(12, 43)(1234)(234)(34)2_x$ is reducible. We are therefore left with 122 concomitants which form the complete system for the two quadratics.

THE JOHNS HOPKINS UNIVERSITY.

Concerning Continuous Curves in Metric Space.*

By W. L. Ayres.

It is the purpose of this paper to prove some results concerning continuous curves lying in a metric space. Most of these results are known for Euclidean space of two dimensions. The proofs for two dimensions do not extend readily since they depend, in general, upon two theorems, neither of which is true in space of more than two dimensions.† Also much of the complexity of the proofs of the theorems of the present paper is due to the failure of these two theorems. Wherever a theorem is known for the two-dimensional case, or for some other special case, we shall refer to this in a footnote. Throughout this paper we assume that all point sets mentioned lie in a metric space which is locally compact in the sense that for each point p and each number $\epsilon > 0$, the set of all points $[x]$ such that $\rho(p, x) \leqq \epsilon$ is a compact point set.

Notations. If K is any point set, $\bar{K}$ denotes the point set consisting of the points of K together with all limit points of K that do not belong to K. If x and y are points, the symbol $\rho(x, y)$ denotes the distance from x to y. If X and Y are point sets, the symbol $\rho(X, Y)$ denotes the greatest lower bound of the set of numbers $\rho(x, y)$ where x is a point of X and y is a point of Y. If xyz denotes an arc with end-points x and y, the symbols $<xzy$, $xzy>$, and $<xzy>$ denote $xzy - x$, $xzy - y$ and $xzy - x - y$, respectively. If x and y are points of an arc α, then the subset xy of α, or the subarc xy of α, denotes the arc with end-points x and y which is a subset of α. If X and Y are point sets, the symbol $X \cdot Y$ denotes the set of all points common to the sets X and Y.

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* Presented to the American Mathematical Society September 10, 1927.

† The two theorems to which we refer are the accessibility theorem of R. L. Wilder and a theorem of the author that there remain only a finite number of maximal connected subsets of diameter greater than $\epsilon > 0$ when a bounded continuous curve is removed from a given continuous curve. See (A) R. L. Wilder, "Concerning Continuous Curves," *Fundamenta Mathematicae*, Vol. 7 (1925), pp. 340-377, theorem 1, and (B) W. L. Ayres, "Note on a Theorem Concerning Continuous Curves," *Annals of Mathematics*, Vol. 28 (1927), pp. 501-502. An example to show that both theorems are false in three dimensions may be found at the end of Theorem 1 of the present paper. To see that the theorem of the author is false, we take $OP + \alpha$ as the bounded continuous curve. Then for any positive number $\epsilon < 2$ there are infinitely many maximal connected subsets of diameter greater than ϵ remaining after the set $OP + \alpha$ is removed.

THEOREM 1. *In order that an M-boundary point P of an M-domain D* * *of a continuous curve M be accessible* † *from D it is sufficient that if C_1 is any hypersphere* ‡ *with center at P there should exist a concentric hypersphere C_2 with radius smaller than that of C_1 such that only a finite number of the components* § *of $D \cdot I(C_1)$ contain points of $I(C_2)$.*

Proof. Let C_1 be a hypersphere of radius 1 with center P. By hypothesis there exists a hypersphere C' with center P and radius less than 1 such that only a finite number of the components of $D \cdot I(C_1)$ have points in $I(C')$. But if C'' is any hypersphere with center P and radius less than that of C', then the above property is satisfied with respect to C''. Hence there exists a hypersphere C_2 with center P and radius less than $\frac{1}{2}$ such that only a finite number of the components of $D \cdot I(C_1)$ have points in $I(C_2)$. Let these be $D_{11}, D_{12}, D_{13}, \cdots, D_{1n_1}$. Since P is a limit point of D there is some integer $k_1 (1 \leqq k_1 \leqq n_1)$ such that P is a limit point of D_{1k_1}. There exists a hypersphere C_3 with center P and radius less than $\frac{1}{3}$ and less than the radius of C_2 such that only a finite number of the components of $D \cdot I(C_2)$ have points in $I(C_3)$. Then there are only a finite number of components of $D_{1k_1} \cdot I(C_2)$ that have points in $I(C_3)$. Let these be $D_{21}, D_{22}, D_{23}, \cdots, D_{2n_2}$. As P is a limit point of D_{1k_1} there is an integer $k_2 (1 \leqq k_2 \leqq n_2)$ such that P is a limit point of D_{2k_2}. In general there exists a hypersphere C_{i+1} with center P and radius less than both $1/i + 1$ and the radius of C_i such that only a finite number of the components of $D \cdot I(C_i)$ have points in $I(C_{i+1})$. Hence there are only a finite number, $D_{i1}, D_{i2}, \cdots, D_{in_i}$, of components of $D_{i-1,\, k_{i-1}} \cdot I(C_i)$ which have points in $I(C_{i+1})$. Hence there is one of these which has P as a limit point. Let D_{ik_i} denote this one.

Let Q be any point of D and let P_1 be any point of D_{1k_1} distinct from Q. There is an arc α_1 of D whose end-points are Q and P_1.¶ Let D_{m_1} be the first

* A connected subset D of a continuum M is said to be an *M-domain* if $M - D$ is closed. The set of all limit points of D that do not belong to D is the M-boundary of D. See reference (A), page 341.

† A boundary point P of a domain D is said to be *accessible* from D if for any point Q of D there exists an arc with end-points P and Q and which lies in D except for the point P.

‡ If p is a point and r is a positive number, the *hypersphere* with center p and radius r is the set of all points $[x]$ of the space such that $\rho(x, p) = r$. If S is a hypersphere, $I(S)$ and $E(S)$ denote the interior and exterior, respectively, of S, i. e. the set of all points $[x]$ of the space such that $\rho(x, p) < r$ and $\rho(x, p) > r$, respectively.

§ A connected subset H of a point set K is said to be a *component* of K if H is not a proper subset of any connected subset of K.

¶ (C) R. L. Moore, "Concerning Continuous Curves in the Plane," *Mathematische Zeitschrift*, Vol. 15 (1922), pp. 254-260, Theorem 1. While Professor Moore proves the theorem for two dimensions only, it is obvious that the same proof holds for our space.

set in the series $D_{1k_1}, D_{2k_2}, \cdots$, which contains no point of α_1. Let P_2 be a point of D_{m_1}. There is an arc α_2 of D_{1k_1} from P_1 to P_2. Let D_{m_2} be the first set in the series $D_{1k_1}, D_{2k_2}, \cdots$, which contains no point of $\alpha_1 + \alpha_2$, and let P_3 be a point of D_{m_2}. There exists an arc α_3 of D_{m_1} with end-points P_2 and P_3. In general let D_{m_i} be the first set of the series $D_{1k_1}, D_{2k_2}, D_{3k_3}, \cdots$ which contains no point of $\sum_{j=1}^{i} \alpha_j$, and let P_{i+1} be a point D_{m_i}. There exists an arc α_{i+1} of $D_{m_{i-1}}$ with end-points P_i and P_{i+1}.

Let p_1 be the first point of α_2 on the arc α_1 in the order from Q to P_1. Let p_2 be the first point of α_3 on the arc α_2 in the order from p_1 to P_2. In general let p_i be the first point of α_{i+1} on the arc α_i in the order from p_{i-1} to P_i. It is easy to see that

$$P + \text{ subarc } Qp_1 \text{ of } \alpha_1 + \sum_{i=2}^{\infty} \text{subarc } p_{i-1}p_i \text{ of } \alpha_i$$

is an arc from Q to P lying in D except for the point P.

The condition of Theorem 1 is by no means necessary and it is probable that much weaker conditions may be found. It is not the purpose of this paper to study accessibility except in so far as may be needed for the remaining theorems. However, it may be of interest to observe that neither of the conditions given by R. L. Wilder * is sufficient in Euclidean space of three dimensions. Consider the following example:

In a Euclidean three-space with a rectangular coördinate system, for every positive integer n let R_n be the set of all points (x, y, z) such that

$$y = nx,\ 0 \leqq y \leqq 1/n,\ 0 \leqq z \leqq 2 - 1/n.$$

In each rectangular set R_n let A_n be the point $(1/n^2, 1/n, 0)$, and let α be the arc consisting of the point $O = (0, 0, 0)$ and the straight line intervals A_1A_2, A_2A_3, A_3A_4, $\cdots$. Let P be the point $(0, 0, 2)$. Let

$$M = P + \alpha + \sum_{i=1}^{\infty} R_i,$$

and let the M-domain D be $M - OP$. Both conditions given by R. L. Wilder are satisfied but the point P of the M-boundary of D is not accessible from D.†

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* See reference (A).

† The accessibility theorem of R. L. Wilder places conditions upon the boundary of the domain only. In this example we have an M-domain whose M-boundary is simply an arc. As an arc is a very simple type of boundary, it seems evident that any general accessibility theorem for M-boundaries of M-domains in space of more than two dimensions must place some conditions on the M-domain itself.

DEFINITIONS. If M is a continuous curve, α is an arc of M, A and B are distinct points of α and D is a subset of $M-\alpha$, the set of arcs K will be said to be a *chain of D from A to B* if (a) K consists of a set of arcs x_1y_1, x_2y_2, $x_3y_3, \cdots, x_my_m$ such that (1) the arc α contains every point x_i and every point y_i, (2) for each i, x_{i+1} lies between x_i and y_i on α but not between y_{i-1} and A, and y_{i+1} lies on the subset $< y_iB$ of α, (3) D contains $< x_iy_i >$ for every i, (4) if $i \neq j$, $<x_iy_i>$ and $<x_jy_j>$ have no points in common, (5) $x_1 = A$ and $y_n = B$; or (b) K consists of a set of arcs x_1y_1, x_2y_2, x_3y_3, $\cdots$ satisfying (1) to (4) above and such that $x_1 = A$ and B is the sequential limit point of $[y_i]$; or (c) K consists of a set of arcs x_0y_0, $x_{-1}y_{-1}$, $x_{-2}y_{-2}, \cdots$ satisfying (1) to (4) above and such that $y_0 = B$ and A is the sequential limit point of $[x_{-i}]$; or (d) K consists of a set of arcs $\cdots, x_{-2}y_{-2}$, $x_{-1}y_{-1}$, x_0y_0, x_1y_1, $x_2y_2, \cdots$ satisfying (1) to (4) above and such that A is the sequential limit point of $[x_i]$ for i negative and B is the sequential limit point of $[x_i]$ for i positive. The chain will be said to be of *type* (a), (b), (c) or (d) according as (a), (b), (c) or (d) is true. A chain will be said to be *convergent* if for any positive number ϵ it contains only a finite number of arcs of diameter greater than ϵ.

THEOREM 2. *If M is a continuous curve, α is an arc of M one of whose end-points is B, D is a component of $M-\alpha$ having B and a point A of α distinct from B as limit points, and ϵ is a positive number, then there is a point A' of α whose distance from A is less than ϵ such that there is a convergent chain of D of type (a) or (b) from A' to B.*

Proof. Let ϵ_1 be any positive number less than both ϵ and $\frac{1}{2}\rho(A,B)$. As M is connected im kleinen at A there exists a positive number δ_{ϵ_1} such that any point of M whose distance from A is less than δ_{ϵ_1} can be joined to A by an arc of M every point of which is at a distance from M less than ϵ_1. There is a point P of D such that $\rho(A,P) < \delta_{\epsilon_1}$. Let α_1 be an arc of M with end-points A and P, every point of which is within a distance ϵ_1 of A. Let A' be the first point of α on α_1 in the order from P to A. Then $\rho(A, A') < \epsilon_1 < \epsilon$.

If B is accessible from D, there is an arc α_2 with end-points P and B and lying in D except for the point B. Let P' be the first point of the subarc PA' of α_1 on α_2 in the order B to P. Then the subarc $P'A'$ of α_1 plus the subarc $P'B$ of α_2 is a chain of D from A' to B.

If B is not accessible from D, by Theorem 1 there exists a hypersphere S with center at B and radius r such that if S' is any hypersphere with center B and radius less than r, then infinitely many of the components of $D \cdot I(S)$

have points in $I(S')$. It is evident that if S'' and S''' are any two hyperspheres with center B such that the radius of S''' is less than the radius of S'', which is less than or equal to r, then infinitely many of the components of $D\cdot I(S'')$ have points in $I(S''')$. Let $[x]$ be any infinite set of points of $D\cdot S'''$ such that no two belong to the same component of $D\cdot I(S'')$. Since $[x]$ is an infinite bounded set, it has at least one limit point y. The point y belongs to $D+\alpha$ since D and M-D-α are mutually separated.* Furthermore, if y belongs to D, it belongs to a component D_y of $D\cdot I(S'')$. But D_y and $D\cdot I(S'')-D_y$ are mutually separated.† Hence y belongs to α. Then every limit point of such a set $[x]$ belongs to α.

Let $S_1, S_2, S_3, \cdots$ be a sequence of hyperspheres with center at B and radii $r_1, r_2, r_3, \cdots$, such that (1) r_1 is the smaller of r and ϵ_1, (2) $r_{i+1} < \frac{1}{2}r_i$ for each i, (3) if x is any point of $\alpha\cdot[S_i+I(S_i)]$ then x lies on the subset $By>$ of α where y is any point of $\alpha\cdot S_{i-1}$. There exists a sequence $D_1, D_2, D_3, \cdots$ of distinct components of $D\cdot I(S_1)$ such that for each i, D_i contains a point z_i in $I(S_i)$. For each i and each $j<i$ let D_{ji} be the component of $D_i\cdot I(S_j)$ which contains z_i.

For each $i>4$ let x_{5i} be a point of $D_i\cdot S_5$. Let v_1 be a limit point of $[x_{5i}]$. We have shown that α contains v_1. Let $\epsilon_{12}=\frac{1}{2}\rho(v_1, S_4+S_6)$. There exists a positive number δ_{12} such that every point of M whose distance from v_1 is less than δ_{12} can be joined to v_1 by an arc of M every point of which is within a distance ϵ_{12} of v_1. There is a point x_{5k_1} such that $\rho(v_1, x_{5k_1})<\delta_{12}$. Let α_4 denote an arc of M with end-points v_1 and x_{5k_1} every point of which is within a distance ϵ_{12} of v_1, and let t_1 be the first point of α on α_4 in the order from x_{5k_1} to v_1. Since D_{k_1} is a component of $D\cdot I(S_1)$ and α_4 is a subset of $I(S_1)$, every point of the subset $x_{5k_1}t_1>$ of α_4 belongs to D_{k_1}. There exists an arc α_3 of D with end-points P and x_{5k_1}. The set consisting of the arc α_3 plus the subarc PA' of α_1 plus the subarc $x_{5k_1}t_1$ of α_4 contains an arc β_1 with end-points A' and t_1 such that D contains $<\beta_1>$.

Only a finite number of the sets $D_{29}, D_{210}, D_{211}, \cdots$, contain points of β_1.‡ For each $i\geqq 9$ let x_{3i} be a point of $D_{2i}\cdot S_3$ and let x_{9i} be a point of $D_{2i}\cdot S_9$. There exists a sequence of integers $i_1, i_2, i_3, \cdots$, such that

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* (D) C. Kuratowski, "Une définition topologique de la ligne de Jordan," *Fundamenta Mathematicae*, Vol. 1 (1920), pp. 40-43; and H. Hahn, "Ueber die Komponenten offener Mengen," *Fundamenta Mathematicae*, Vol. 2 (1921), pp. 189-192.

† See reference (D).

‡ Since β_1 contains only a finite number of mutually exclusive subarcs containing a point of S_1 and a point of S_2.

(1) $[x_{3i_n}]$ has a sequential limit point u_2 on $\alpha \cdot S_8$,
(2) $[x_{9i_n}]$ has a sequential limit point v_2 on $\alpha \cdot S_9$,
(3) no set D_{2i_n} contains a point of β_1.

Let $\epsilon_{21} = \frac{1}{2}\rho(u_2, S_2 + S_4)$ and $\epsilon_{22} = \frac{1}{2}\rho(v_2, S_8 + S_{10})$. There exist positive numbers δ_{21} and δ_{22} such that any point of M whose distance from $\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}$ is less than $\begin{Bmatrix} \delta_{21} \\ \delta_{22} \end{Bmatrix}$ can be joined to $\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}$ by an arc of M every point of which is within a distance $\begin{Bmatrix} \epsilon_{21} \\ \epsilon_{22} \end{Bmatrix}$ of $\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}$. Since u_2 and v_2 are sequential limit points of $[x_{3i_n}]$ and $[x_{9i_n}]$, there is a number k_2 (which is one of the numbers i_n) such that $\rho(u_2, x_{3k_2}) < \delta_{21}$ and $\rho(v_2, x_{9k_2}) < \delta_{22}$. Let α_5 and α_6 be arcs of M with end-points u_2 and x_{3k_2} and v_2 and x_{9k_2} respectively such that every point of α_5 is within a distance ϵ_{21} of u_2 and every point of α_6 is within a distance ϵ_{22} of v_2. Let s_2 and t_2 be the first points of α on α_5 and α_6 in the orders from x_{3k_2} to u_2 and from x_{9k_2} to v_2 respectively. Since D_{2k_2} is a component of $D \cdot I(S_2)$ the subsets $x_{3k_2}s_2 >$ of α_5 and $x_{9k_2}t_2 >$ of α_6 belong to D_{2k_2}. Let α_7 be an arc of D_{2k_2} with end-points x_{3k_2} and x_{9k_2}. Then the set composed of the arc α_7 plus the subarc $x_{3k_2}s_2$ of α_5 plus the subarc $x_{9k_2}t_2$ of α_6 contains an arc β_2 with end-points s_2 and t_2 and such that D_{2k_2} contains $< \beta_2 >$. Since D_{2k_2} contains no point of β_1, the arcs β_1 and β_2 have no common points. From the manner in which the hyperspheres were chosen we have the order $A's_2t_1t_2B$ on α.

Repeating the above process with the sets $D_{6,13}, D_{6,14}, D_{6,15}, \cdots$, we obtain an arc β_3 whose end-points are s_3 and t_3 such that (1) β_3 is a subset of $I(S_6)$, (2) s_3 is a point of α whose distance from S_7 is less than $\rho(S_7, S_6 + S_8)$ and t_3 is a point of α whose distance from S_{13} is less than $\rho(S_{13}, S_{12} + S_{14})$, (3) β_3 has no point in common with β_1 or β_2 and D contains $< \beta_3 >$, (4) on the arc α we have the order $A's_2t_1s_3t_2t_3B$.

Let us continue this process indefinitely. We obtain an infinite sequence of arcs $\beta_1, \beta_2, \beta_3, \beta_4, \cdots$, such that (1) β_i has end-points s_i and t_i * on α in the order $s_1s_2t_1s_3t_2s_4t_3s_5 \cdots B$, (2) for each $i > 1$, β_i is a subset of $I(S_{4i-6})$ and hence the diameter of β_i approaches zero as i increases indefinitely, (3) the point B is the sequential limit point of $[t_i]$, (4) for every i, D contains $< \beta_i >$ and if $i \neq j$ then $\beta_i \cdot \beta_j = 0$, (5) $\rho(A, A') < \epsilon$. Then $\beta_1, \beta_2, \beta_3, \cdots$, is the desired convergent chain of D of type (b) from A' to B.

* Let s_1 be the point A'.

THEOREM 3.* *In order that a point P be an end-point † of a continuous curve M it is necessary and sufficient that P be a non-cut-point ‡ of M which lies on no simple closed curve of M.*

Proof. The proof given by the author § for the necessity of the condition in two dimensions holds equally well in the space considered here.

The condition is sufficient. Let P be a point of M satisfying the condition and suppose that P is not an end-point of M. Then there exists an arc α of M one of whose end-points is P, such that $M - (\alpha - P)$ contains a connected set N containing more than one point and containing P. Let

$$M - P = (\alpha - P) + M_1.$$

Then M_1 contains $N - P$. Let q be any point of $N - P$ and let D_q be the component of $M - \alpha$ containing q. As N is connected, one of the sets $N \cdot D_q$ and $N - N \cdot D_q$ contains a limit point of the other. The set $N \cdot D_q$ cannot contain a limit point of $N - N \cdot D_q$ since D_q is an open subset of M.¶ Since D_q is a component of $M - \alpha$, no point of $N - N \cdot D_q - P$ is a limit point of $N \cdot D_q$. Hence P is a limit point of $N \cdot D_q$ and consequently of D_q. Since P is a non-cut-point of M, one of the sets $(\alpha - P + M_1 - D_q)$ and D_q must contain a limit point of the other. As $M - D_q$ is closed, the set $(\alpha - P + M_1 - D_q)$ must contain a limit point of D_q. But every limit point of D_q which does not belong to D_q belongs to α. Hence $\alpha - P$ contains a limit point A of D_q.

Let ϵ be a positive number less than $\rho(A, P)$. By Theorem 2 there is a point A' of α such that $\rho(A, A') < \epsilon$ and a convergent chain of $M - \alpha$ from A' to P. This chain contains just one arc or is of type (b). If the chain contains just one arc β_1, then β_1 plus the subarc $A'P$ of α is a simple closed curve of M containing P. But this is a contradiction as P lies on no simple closed curve of M.

If the chain is of type (b), let $\beta_1, \beta_2, \beta_3, \cdots$, be the arcs of the chain.

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* For the theorem in two dimensions, see (E) W. L. Ayres, "Concerning continuous curves and correspondences," *Annals of Mathematics*, Vol. 28 (1927), pp. 396-418, Theorem 3.

† A point P of a continuous curve M is said to be an *end-point* of M if for any arc α of M one of whose end-points is P, the set $M - (\alpha - P)$ contains no connected subset containing more than one point and containing P. See reference (A), page 358.

‡ A point P of a connected set M is said to be a *non-cut-point* or a *cut-point* of M according as $M - P$ is connected or not.

§ See reference (E).

¶ See reference (D).

Let

$$\alpha_1 = P + \sum_{i=1}^{\infty} \beta_{2i-1} + \sum_{i=1}^{\infty} \text{subarc } t_{2i-1}s_{2i+1} \text{ of } \alpha,$$

$$\alpha_2 = P + \text{subarc } A's_2 \text{ of } \alpha + \sum_{i=1}^{\infty} \beta_{2i} + \sum_{i=1}^{\infty} \text{subarc } t_{2i}s_{2i+2} \text{ of } \alpha.$$

Then α_1 and α_2 are arcs of M from A' to P which have only these two points in common. Hence $\alpha_1 + \alpha_2$ is a simple closed curve of M containing P. But by hypothesis P belongs to no simple closed curve of M.

As a corollary of Theorem 3 we have a theorem which was proved by G. T. Whyburn * for two dimensions.

THEOREM 4. *Every continuous curve is the sum of its end-points, cut-points and simple closed curves.*

THEOREM 5.† *In order that a point P of a continuous curve M should be an end-point of M it is necessary and sufficient that no arc of M should have P as an interior point.*

Proof. The proof of the necessity of the condition given for two dimensions holds without regard to the containing space.

The condition is sufficient. Suppose P is a point of M that is interior to no arc of M and that P is not an end-point of M. Then, by Theorem 3, the point P is either a cut-point of M or lies on some simple closed curve of M. If P lies on a simple closed curve J of M, let A and B be two points of J which are distinct and distinct from P. Then there is an arc of J with end-points A and B that contains P as an interior point, contrary to the assumed condition. If P is a cut-point of M, let $M - P = M_1 + M_2$, where M_1 and M_2 are non-vacuous mutually separated sets. Let A be a point of M_1 and let B be a point of M_2. Any arc of M with end-points A and B must have P as an interior point. Hence the assumption that P is not an end-point of M leads to a contradiction with the given condition.

Theorems 5 and 6 of my paper, "Concerning Continuous Curves and Correspondences", ‡ hold in our general space without change in proof since the proofs given depend only on Theorem 5 of the present paper. Theorems 7 to 15 inclusive of my paper, "Concerning Continuous Curves

* (F) G. T. Whyburn, "Concerning Continua in the Plane," *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 369-400, Theorem 22.

† For the theorem in two dimensions, see reference (F), Theorem 12. See also reference (E), Theorem 4.

‡ See reference (E). See also reference (F), Theorem 21. However, the proof given by Whyburn is strictly two-dimensional.

and Correspondences", extend to the space considered here with the same proofs since they concern internal properties of continuous curves containing at most one simple closed curve, and every such curve is in continuous (1, 1) correspondence with some curve of the same type in the plane.*

THEOREM 6.† *If K_1 and K_2 are mutually exclusive closed subsets of the continuous curve M and no point of M separates* ‡ *K_1 and K_2 in M, then there are two arcs α_1 and α_2 of M such that* (1) *for each i* ($i = 1, 2$), *α_i has a point of K_1 as one end-point and a point of K_2 as the other end-point and $M - K_1 - K_2$ contains $< \alpha_i >$,* (2) *$< \alpha_1 >$ and $< \alpha_2 >$ are mutually exclusive.*

Proof. Let u be a point of K_1 and v be a point of K_2, and let α' be an arc of M with end-points u and v. In the order from u to v let x be the last point of K_1 on α', and let y be the first point of K_2 on the subarc xv of α'. Let α denote the subarc of α' with end-points x and y. On α we will define order as being from x to y. If any component of $M - \alpha$ contains points of both K_1 and K_2, then this component contains an arc having a point of K_1 as one end-point and a point of K_2 as the other end-point and having no other point in common with $K_1 + K_2$. Then this arc together with the arc α forms a pair of arcs having all the properties of our theorem. If no component of $M - \alpha$ contains points of both K_1 and K_2, then for any

* For the case of a continuous curve containing no simple closed curve, see T. Ważewski, "Sur les courbes de Jordan ne renfermant aucune courbe fermée de Jordan," *Annales de la Société Polonaise de Mathématique*, Vol. 2 (1923), pp. 49-170, K. Menger, "Über reguläre Baumkurven," *Mathematische Annalen*, Vol. 96 (1926), pp. 572-582, and H. M. Gehman, "Concerning Acyclic Continuous Curves," *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 553-568. The case where the continuous curve contains just one simple closed curve is not essentially different.

† For the case where M lies in a plane and K_1 and K_2 are single points, this theorem has been proved by G. T. Whyburn, "Some Properties of Continuous Curves," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 305-308, Theorem III. The same special case was announced to the American Mathematical Society February 26, 1927, by the author, but it has never been published. See an abstract, "On the Separation of Points of a Continuous Curve by Arcs and Simple Closed Curves," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), p. 266. Theorems closely related to Theorem 6 have been stated by K. Menger, "Zur allgemeinen Kurventheorie," *Fundamenta Mathematicae*, Vol. 10 (1926), pp. 96-115, Statz β; and N. E. Rutt, "Concerning the cut-points of a continuous curve when the arc-curve *ab* contains exactly *n* independent arcs," *American Journal of Mathematics*, Vol. 51 (1929), pp. 217-246.

‡ If K_1 and K_2 are mutually exclusive subsets of a connected set M, a subset H of M is said to separate K_1 and K_2 in M if $M - H$ is the sum of two mutually separated sets, one containing K_1 and the other containing K_2.

component d of $M - \alpha$ let P_{xd} and P_{yd} denote the first and last limit points of d on α. Let H_1 denote the set of all points $P_{yd'}$, where d' is a component of $M - \alpha$ containing a point of K_1, and let H_2 denote the set of all points $P_{xd''}$, where d'' is a component of $M - \alpha$ containing a point of K_2. Let x' denote the last point of $\bar{H}_1 + x$ on α, and let y' denote the first point of $\bar{H}_2 + y$ on α.

If $x' \neq x$ and ϵ is any positive number, there is a point $P_{yd'}$ of H_1 such that $\rho(x', P_{yd'}) < \epsilon/2$. There exists a positive number δ_ϵ such that $\delta_\epsilon < \rho(P_{yd'}, K_1)$ and any point of M whose distance from $P_{yd'}$ is less than δ_ϵ can be joined to $P_{yd'}$ by an arc of M every point of which is within a distance $\epsilon/2$ of $P_{yd'}$. There is a point z of d' such that $\rho(z, P_{yd'}) < \delta_\epsilon$. Let γ_1 be an arc of M with end-points z and $P_{yd'}$, every point of which is within a distance $\epsilon/2$ of $P_{yd'}$. In the order from z to $P_{yd'}$ let x'' be the first point of α on γ_1. The component d' contains a point w of K_1, and let γ_2 be an arc of d' with end-points w and z. In the order from w to z let w' be the last point of K_1 and let z' be the first point of the subarc zx'' of γ_1 on the subarc $w'z$ of γ_2. The subarc $w'z'$ of γ_2 plus the subarc $z'x''$ of γ_1 is an arc of M having one end-point in K_1 and one end-point on α within a distance ϵ of x' and lying except for its end-points in $M - (K_1 + K_2 + \alpha)$. Similarly if $y' \neq y$, there is an arc having the same properties with respect to y' and K_2.

Now suppose y' precedes x' on α. There exists a positive number ϵ such that if y'' and x'' are points of α so that $\rho(x', x'') < \epsilon$ and $\rho(y', y'') < \epsilon$, then y'' precedes x'' on α. Since $x' \neq x$ and $y' \neq y$, there exist two mutually exclusive arcs γ_3 and γ_4 such that (1) γ_3 has a point of K_2 as one end-point and a point y'' of α such that $\rho(y', y'') < \epsilon$ as the other end-point, (2) γ_4 has a point of K_1 as one end-point and a point x'' of α such that $\rho(x', x'') < \epsilon$ as the other end-point, (3) $M - (K_1 + K_2 + \alpha)$ contains $< \gamma_3 >$ and $< \gamma_4 >$. Then the arc composed of γ_3 plus the subarc xy'' of α together with the arc composed of γ_4 plus the subarc yx'' of α form a pair of arcs satisfying the conditions of our theorem. This completes the proof for the case where y' precedes x' on α and for the remainder of the argument we will suppose that y' does not precede x' on α.

If $x' = y$, then either (1) y is a point of H_1 or (2) y belongs to $\bar{H}_1 - H_1$. In the first case there is a component d' of $M - \alpha$, containing a point q of K_1 such that $P_{yd'} = y$. If y is accessible from d' there exists an arc β whose end-points are q and y and which lies in d' except for y. In the order from q to y let q' be the last point of K_1 on β. Then α together with the subarc $q'y$ of β forms a set of two arcs satisfying the conditions of the theorem. If y is not accessible from d', by Theorem 2, there exists a convergent chain of

d' of type (b) from some point A' of α to y. Let $\beta_1, \beta_2, \beta_3, \cdots$, be the arcs of this chain. Let s_i be the first end-point of β_i on the arc α, and let t_i be the other end-point of β_i. There exists a smallest positive integer k such that for $i \geqq k$, the arc β_i contains no point of the set K_1. There exists a positive number ϵ such that if z is any point of α so that $\rho(z, t_k) < \epsilon$, then z lies on the subset $< s_2y$ of α. There is a positive number δ_ϵ such that any point of M whose distance from t_k is less than δ_ϵ can be joined to t_k by an arc of M every point of which is within a distance ϵ of t_k. If $k > 1$, let $m = k$ and let h be the first point of K_1 on the arc β_{k-1} in the order from t_{k-1} to s_{k-1}. Let ζ denote the subarc ht_{k-1} of β_{k-1}, and let t_{k-1} be denoted by g. If $k = 1$, let c be a point of d' such that $\rho(c, t_1) < \delta_\epsilon$. Let ζ_1 be an arc of d' with end-points c and q, and let ζ_2 be an arc of M with end-points c and t_1 such that every point of ζ_2 is within a distance ϵ of t_1. The set $\zeta_1 + \zeta_2$ contains an arc λ which has a point h of K_1 as one end-point and a point e of α as the other end-point and no other point in common with $K_1 + \alpha$. If λ has no point in common with $\sum_{i=1}^{\infty} < \beta_i >$, let $g = e$, $\zeta = \lambda$, and let m be the smallest integer such that e lies between s_m and t_m on α. If λ has a point in common with $\sum_{i=1}^{\infty} < \beta_i >$, in the order from h to e let b be the first point of $\sum_{i=1}^{\infty} < \beta_i >$ on the arc λ. It is easy to prove the existence of a first point. Let β_{m-1} be the arc containing b, and let $t_{m-1} = g$ and ζ be the arc composed of the subarc hb of λ plus the subarc bt_{m-1} of β_{m-1}. Now in any of the preceding possibilities of case (1) let

$$\alpha_1 = \text{subarc } xs_m \text{ of } \alpha + \sum_{i=0}^{\infty} \beta_{m+2i} + \sum_{i=0}^{\infty} \text{subarc } t_{m+2i}s_{m+2i+2} \text{ of } \alpha,$$

$$\alpha_2 = \zeta + \text{subarc } gs_{m+1} \text{ of } \alpha + \sum_{i=0}^{\infty} \beta_{m+2i+1} + \sum_{i=0}^{\infty} \text{subarc } t_{m+2i+1}s_{m+2i+3} \text{ of } \alpha,$$

Then α_1 and α_2 are two arcs satisfying the conditions of our theorem.

In case (2) there exists a set of components $d_1, d_2, \cdots$, of $M - \alpha$ such that each component contains a point of K_1 and each contains a point p_i such that $\rho(p_i, y) < 1/i$. Let S be a hypersphere with center y such that every point of K_1 lies in $E(S)$. For each i let d'_i be the component of $I(S) \cdot d_i$ that contains p_i. By methods almost identical with those of Theorem 2 we may show that there exists a convergent chain of $[d'_i]$ of type (b) from some point of $< \alpha >$ to y. With the use of this chain we may select two arcs satisfying the hypothesis of our theorem as in the above paragraph.

This completes the argument for the case where $x' = y$. Similarly we may complete the proof if $y' = x$. For the remainder of our proof we shall consider $x' \neq y$ and $y' \neq x$.

For each point P of $<\alpha>$ let G_P denote the set of all points P_{xd} for all components d of $M-\alpha$ such that P lies between P_{xd} and P_{yd} on α. If $x'=y'$, let $Q=x'$. If $x'\neq y'$ let Q be some point of the subset $<x'y'>$ of α. Let T denote the point x if $x'=x$ and if $x'\neq x$ let it denote the subset $xx'>$ of α. Let Q_1 be the first point of $\bar{G}_Q$. If Q_1 belongs to T, then either (1) there is a component D of $M-\alpha$ which has a point of T and a point of α between Q and y as limit points, or (2) there is no such component. If we have case (1), by Theorem 2 there is a convergent chain of D from some point of α between y and Q to some point of T. If we have case (2), we must have $x'=x$; and if S_1 is a hypersphere with center x and radius sufficiently small that the subarc yQ of α lies in $E(S_1)$ and if S_2 is any hypersphere concentric with and interior to S_1, there are infinitely many components of $M-\alpha$ which have points in $I(S_2)$ and $E(S_1)$. In exactly the same manner as in the proof of Theorem 2 we may obtain a convergent chain in this situation.

If Q_1 does not belong to T, let ϵ_1 be a positive number less than 1 such that if z is any point of the subarc yQ_1 of α so that $\rho(z,Q_1)<\epsilon_1$, then there is a component d of $M-\alpha$ such that both z and Q_1 lie between P_{xd} and P_{yd}. There is a point s_1 of G_Q such that $\rho(s_1,Q_1)<\epsilon_1/2$. Let d_1 be a component of $M-\alpha$ such that $P_{xd_1}=s_1$ and P_{yd_1} lies on the subset $yQ>$ of α. There exists a number $\delta_1>0$ such that any point of M whose distance from s_1 is less than δ_1 can be joined to s_1 by an arc of M every point of which is within a distance $\epsilon_1/2$ of s_1. Let p_1 be a point of d_1 so that $\rho(p_1,s_1)<\delta_1$, and let α_1 denote an arc of M from p_1 to s_1 every point of which is within a distance $\epsilon_1/2$ of s_1. Let u_1 be the first point of α on α_1 in the order from p_1 to s_1. Then $\rho(u_1,Q_1)<\epsilon_1$.

The component d_1 has a limit point t_1 on the subset $yQ>$ of α. There is a point p_2 of d_1 and an arc α_2 of M from p_2 to t_1 such that every point of α_2 is at a distance from t_1 less than $\rho(t_1$, subarc xQ of $\alpha)$. Let v_1 be the first point of α on the arc α_2 in the order p_2 to t_1. There is an arc α_3 of d_1 with end-points p_1 and p_2. The set $\alpha_1+\alpha_2+\alpha_3$ contains an arc β_1 with end-points u_1 and v_1 and such that d_1 contains $<\beta_1>$. It is easy to see that on α we have the order $xQ_1u_1Qv_1y$.

Let Q_2 be the first point of the set $\bar{G}_{u_1}$ on α. If Q_2 belongs to T, our argument is completed as above. If Q_2 does not belong to T, let ϵ_2 be a positive number less than ½ such that if z is any point of the subset $<Q_2y$ of α such that $\rho(z,Q_2)<\epsilon_2$, then there is a component d of $M-\alpha$ such that both z and Q_2 lie between P_{xd} and P_{yd} on α. Just as above we may show that there is a component d_2 of $M-\alpha$ different from d_1 and an arc β_2 with end-points u_2 and v_2 of α such that $\rho(u_2,Q_2)<\epsilon_2$, d_2 contains $<\beta_2>$, and on α we have the order $xQ_2u_2Q_1u_1v_2v_1$.

Let Q_3 be the first point of the set $\bar{G}_{u_2}$ on α. If Q_3 belongs to T, our argument is completed as above. If Q_3 does not belong to T, let ϵ_3 be a positive number less than 1/3 such that if z is any point of the subarc yQ_3 of α so that $\rho(z, Q_3) < \epsilon_3$, then there is a component d of $M - \alpha$ such that both z and Q_3 lie between P_{xd} and P_{yd}. There exists a component d_3 of $M - \alpha$ different from both d_1 and d_2 and an arc β_3 with end-points u_3 and v_3 of α such that $\rho(u_3, Q_3) < \epsilon_3$, d_3 contains $< \beta_3 >$, and we have the order $xQ_3u_3Q_2u_2v_3u_1v_2v_1y$ on α.

Continue this process indefinitely unless for some n, the point Q_n belongs to T. In this case we may complete the argument as above showing that there is a convergent chain of $M - \alpha$ joining v_1 and some point of T. If the process continues indefinitely, the points $u_1, u_2, u_3, \cdots$, approach a point x_1 as a sequential limit point. If $x_1 \neq x$, there is a component d of $M - \alpha$ such that x_1 lies between P_{xd} and P_{yd}. Let j be the smallest integer so that u_j lies on the subset $x_1P_{yd} >$ of α. Then the point Q_{j+1} lies on the subarc xP_{xd} of α. And u_{j+2} precedes Q_{j+1} on the arc α. But this is impossible for x_1 precedes every point u_i and P_{xd} precedes x_1. Therefore $x_1 = x$ and the set $\beta_1, \beta_2, \beta_3, \cdots$, is a chain of $M - \alpha$ from v_1 to x.

We shall now show that the chain is convergent. Suppose there exists a positive number η and an increasing sequence of integers $n_1, n_2, n_3, \cdots$, such that β_{n_i} is of diameter greater than η for every positive integer i. Let S be a hypersphere of radius $\eta/2$ with x as a center, and for each value of i let z_{n_i} be a point of $S \cdot \beta_{n_i}$. There exists a point z' and a subsequence $k_1, k_2, k_3, \cdots$, of the sequence $n_1, n_2, n_3, \cdots$, such that z' is the sequential limit point of $[z_{k_i}]$. Since no two of the sets $< \beta_{k_i} >$ lie in the same component of $M - \alpha$, it is easy to see that the arc α contains z'. On the subset $xz' >$ of α let u_j be the last point of the set $[u_i]$ in the order from x to z'. Let γ be the smaller of the numbers $\rho(z'$, subarc xu_j of $\alpha)$ and $\rho(x$, subarc yQ_{j+1} of $\alpha)$. There exists a number $\delta_\gamma > 0$ such that any point of M whose distance from z' or x is less than δ_γ can be joined to z' or x, as the case may be, by an arc of M every point of which is within a distance γ of z' or x. There is an arc β_{k_m} containing points z_{k_m} and w_k such that $\rho(z', z_{k_m}) < \delta_\gamma$ and $\rho(x, w_{k_m}) < \delta_\gamma$. Then the component of $M - \alpha$ containing $< \beta_{k_m} >$ has limit points on α preceding Q_{j+1} and following u_j. But this is contrary to the definition of Q_{j+1}. Thus $\beta_1, \beta_2, \beta_3, \cdots$, is a convergent chain of $M - \alpha$ of type (b) from v_1 to x.

If $y' = y$, let U be the point y. If $y' \neq y$, let U be the subset $< y'y$ of α. In exactly the same manner as above we determine a convergent chain of $M - \alpha$ of type (a) or (b) from a point of α preceding Q to a point of the set U. From the two chains we may select a convergent chain of $M - \alpha$ from

a point of T to a point of U by omitting a finite number of arcs. From this convergent chain, the arc α, and, if $x' \neq x$, an arc of M one of whose end-points belongs to K_1 and the other is a point of α sufficiently close to x' and which has no other point in common with $K_1 + K_2 + \alpha$, and, if $y' \neq y$, an arc of M one of whose end-points is a point of K_2 and the other a point of α sufficiently close to y' and having no other point in common with $K_1 + K_2 + \alpha$, we may select two arcs which satisfy the conditions of the theorem as we did in case (1) of $x' = y$.

THEOREM 7. *If x and y are distinct points of a continuous curve M, then either there is a point of M which separates x and y in M or there is a simple closed curve of M containing both x and y.*

This theorem is a corollary of Theorem 6.

THEOREM 8. *If K_1 and K_2 are two mutually exclusive closed subsets of the continuous curve M and for every point P of M, the sets $(K_1 + P) - P$ and $(K_2 + P) - P$ are non-vacuous and are not separated in M by the point P, then there exist two mutually exclusive arcs α_1 and α_2 of M such that each arc has a point of K_1 as one end-point and a point of K_2 as the other and, except for these points, lies entirely in $M - K_1 - K_2$.*

THEOREM 9. *If K_1 and K_2 are mutually exclusive closed subsets of the continuous curve M and for every point P of $M - K_2$, the sets $(K_1 + P) - P$ and K_2 are non-vacuous and are not separated in M by the point P, then there exist two arcs α_1 and α_2 such that* (1) *each arc has a point of K_1 as one end-point and a point of K_2 as the other and lies, except for these points, entirely in $M - K_1 - K_2$,* (2) *the arcs α_1 and α_2 have no point in common except possibly their end-points that belong to K_2.*

The proof given for Theorem 6 is sufficient to prove Theorems 8 and 9 since the two arcs constructed in proving Theorem 6 have in common their end-points that belong to K_1 only if $x' = x$, and the end-points belonging to K_2 are identical only if $y' = y$.

THEOREM 10.* *In order that a continuous curve be cyclicly connected it is necessary and sufficient that it contain no cut-point.*

Proof. The proof of the necessity of the condition given for two dimensions holds equally well in the space considered here.

* For the result in two dimensions, see (G) G. T. Whyburn, "Cyclicly Connected Continuous Curves," *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 31-38, Theorem 1. A continuous curve M is said to be cyclicly connected if every two points of M lie together on some simple closed curve belonging to M.

The condition is sufficient. Let x and y be any two distinct points of M. Since M contains no cut-point, there is no point of M which separates x and y in M. Then, from Theorem 7, there is a simple closed curve of M containing x and y. Hence M is cyclicly connected.

Theorems 2-7 and 9 of G. T. Whyburn's paper, "Cyclicly Connected Continuous Curves," * hold in n dimensions as the proofs given by Whyburn depend only on our theorem 10 (Whyburn's theorem 1), two theorems of G. T. Whyburn,† and a theorem of R. L. Moore,‡ all of which are true in the space considered here. Also theorem 8 of Whyburn's paper * is true in our space but a different proof of part (2) is necessary to avoid the Wilder accessibility theorem.

THEOREM 11.§ *If the point x of a continuous curve M lies on no simple closed curve of M and α is any arc of M whose end-points are x and any other point z of M, then x is a limit point of the points of M which lie on α and separate x and z in M.*

Proof. Suppose the theorem is not true. Then there is an arc α of M whose end-points are x and some other point z of M, and a positive number ϵ such that no point of α whose distance from x is less than ϵ separates x and z in M. There is a point y of α such that every point of the subarc xy of α is at a distance from x less than ϵ. Hence no point of the subarc xy of α separates x and y in M. Moreover if there is any point of M which separates x and y in M, it must belong to every arc of M with end-points x and y, and thus to the subarc xy of α. Then by Theorem 7 there is a simple closed curve of M containing both x and y. But by hypothesis x belongs to no simple closed curve of M.

THEOREM 12.¶ *If K is any subset of the continuous curve M and $M(K)$ denotes the arc-curve of K with respect to M,‖ then the point set $\overline{M(K)} - M(K)$ is a subset of the point set $\bar{K} - K$.*

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* See reference (G).

† See reference (F), theorems 15 and parts 1) and 3) of 24. Theorems 24 and theorems 10 and 23 of (F), on which theorems 15 and 24 are based, require different proofs in our more general space.

‡ See reference (C), Theorem 1.

§ For the two-dimensional case, see (H) W. L. Ayres, "On the Structure of a Plane Continuous Curve," *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 749-754.

¶ For the theorem in two dimensions, see (I) W. L. Ayres, "Concerning the Arc Curves and Basic Sets of a Continuous Curve," *Transactions of the American Mathematical Society*, Vol. 30 (1928), pp. 567-578, Theorem 2.

‖ The set of all points [P] such that P lies on some arc of M whose end-points belong to K is called the *arc-curve of K with respect to M*. If K contains just one point, we define the arc-curve of K with respect to M as being this single point.

Proof. Suppose $\overline{M(K)} - M(K)$ contains a point x which does not belong to $\bar{K} - K$. Since $M(K)$ contains K, the point x belongs to $M - \bar{K}$. Let z be a point of K and let xz be an arc of M with end-points x and z. In the order from x to z let y be the first point of $\bar{K}$ on xz. Then $x \neq y$. Let α be the subarc of xz with end-points x and y. Since x belongs to $\overline{M(K)} - M(K)$, the set K contains more than one point. For each component d of $M - \alpha$ containing a point of K let P_{xd} denote the first limit point of d on α in the order from x to y. Let H denote the set of all such points P_{xd} for all such components d of $M - \alpha$. Let w be the first point of $\bar{H} + y$ on the arc α in the order x to y. If $w = y$, then

$$M - y = M_1 + M_2,$$

where M_1 and M_2 are non-vacuous mutually separated sets containing x and $\bar{K} - y$ respectively. Then $M(K)$, and thus $\overline{M(K)}$, is a subset of $M_2 + y$. Then x cannot be a point of $\overline{M(K)}$, contrary to hypothesis. Hence $w \neq y$.

In much the same way we may show that there is no point P which separates x and $\bar{K}$ in M. Now let P be a point of $\bar{K}$. If $P \neq y$, then $(\bar{K} + P) - P$ is non-vacuous since it contains y, and P does not separate x and $(\bar{K} + P) - P$ in M since the arc α does not contain P and joins x and a point y of $\bar{K}$. If $P = y$, then $\bar{K} - P$ is non-vacuous and P does not separate x and $\bar{K} - P$ in M since there is a component d of $M - \alpha$ containing a point of $\bar{K}$ which has a limit point u on the subset $wy >$ of α, and the subarc xu of α plus the set d is a connected subset of $M - P$ containing x and a point of K. Thus the hypothesis of Theorem 9 is satisfied and there exist two arcs α_1 and α_2 of M having only the point x in common such that each arc has x as one end-point and a point of $\bar{K}$ as the other end-point and lies in $M - \bar{K} - x$ except for these points. Let y_1 and y_2 be the end-points of α_1 and α_2 belonging to $\bar{K}$. If y_1 belongs to K, let t_1, y_1' and α_3 all be the point y_1. If y_1 belongs to $\bar{K} - K$, let $\epsilon_1 = \rho(y_1, \alpha_2)/2$. There is a positive number δ_1 such that any point of M whose distance from y_1 is less than δ_1 can be joined to y_1 by an arc of M of diameter less than ϵ_1. There is a y_1' of K such that $\rho(y_1', y_1) < \delta_1$ and let α_3 be an arc of M with end-points y_1 and y_1' and of diameter less than ϵ_1. In the order from y_1' to y_1 let t_1 be the first point of α_1 on α_3. In a similar manner we define y_2', t_2 and α_4. Then the sets composed of the subarc $y_1't_1$ of α_3 plus the subarc xt_1 of α_1 plus the subarc xt_2 of α_2 plus the subarc t_2y_2' of α_4 is an arc of M containing x and with the points y_1' and y_2' of K as end-points. Then, by definition, the point x belongs to $M(K)$. But we assumed that $\overline{M(K)} - M(K)$ contains the point x.

Theorems 1, 3 to 9 and 11 to 13 inclusive, and part (1) of 10 of my

paper, "Concerning the Arc-Curves and Basic Sets of a Continuous Curve",* hold in our present space without change in proof since they depend only on theorems 3, 5 and 12 of the present paper, theorems 2 and 3 of G. T. Whyburn's paper, "Cyclicly Connected Continuous Curves",† and a theorem of my paper, "Continuous Curves which are Cyclicly Connected",‡ all of which hold in this space. Part (2) of theorem 10 holds also in this space but the proof must be modified slightly to avoid using the accessibility theorem of Wilder.

THEOREM 13.§ *If x is a point of a continuous curve M and C_x denotes the set consisting of x and all points $[y]$ such that M contains a simple closed curve containing both x and y, then C_x is a continuous curve.*

Proof. Obviously C_x is connected. Let z be any limit point of C_x. If there is any point P which separates x and z in M, then

$$M - P = M_x + M_z,$$

where M_x and M_z are mutually separated sets containing x and z respectively. As z is a limit point of C_x, the set M_z contains a point y of C_x. There is a simple closed curve J of M containing x and y. Both of the arcs of J with end-points x and y cannot contain the point P. But every connected set containing a point of M_x and a point of M_z must contain the point P. Hence there is no point of M which separates x and z in M. Then by Theorem 7 there is a simple closed curve of M containing both x and z. Therefore z belongs to C_x by definition. Then every limit point of C_x belongs to C_x. We may prove that C_x is connected im kleinen as in the two-dimensional case.

THEOREM 14.¶ *Under the hypothesis of Theorem 13, if K is a component of $M - C_x$, then K has just one limit point in C_x.*

Suppose K has two limit points p and q in C_x. Let $\epsilon = \frac{1}{2}\rho(p, q)$. There exists a positive number δ_ϵ such that any point of M whose distance from p or q is less than δ_ϵ can be joined to p or q, as the case may be, by an arc of M of diameter less than ϵ. The set K contains two points u and v such that

$$\rho(u, p) < \delta_\epsilon \quad \text{and} \quad \rho(v, q) < \delta_\epsilon.$$

* See reference (I).

† See reference (G).

‡ *Bulletin de l'Académie Polonaise des Sciences et des Lettres* (1928), pp. 127-142. The theorem referred to is: If x, y and z are distinct points of a cyclicly connected continuous curve, there exists an arc belonging to the continuous curve, containing z and having x and y as end-points.

§ For the theorem in two dimensions, see reference (H), Theorem 2.

¶ For the result in two dimensions see reference (H), Theorem 3.

Let α_1 and α_2 denote arcs of M of diameter less than ϵ with end-points u and p and v and q respectively. Let s and t be the first points of C_x on the arcs α_1 and α_2 in the orders from u to p and from v to q respectively. The set K contains an arc α_3 with end points u and v. The set composed of α_3 plus the subarc su of α_1 plus the subarc tv of α_2 contains an arc α_4 with end-points s and t and such that K contains $< \alpha_4 >$. By an argument following that used in the two-dimensional case of Theorem 13, we may show that every point of an arc whose end-points belong to C_x is a point of C_x. But $M - C_x$ contains the set K and $< \alpha_4 >$ is a subset of K.

THEOREM 15.* *If the continuous curve M contains a cut-point and a simple closed curve, it contains a cut-point which lies on a simple closed curve of M.*

Proof.† Let P be a cut-point of M and let J be a simple closed curve of M. There is a maximal cyclic curve C of M containing J.‡ If C contains the point P, there is a simple closed curve of C containing the point P. If C does not contain P, let R be the component of $M - C$ containing the point P. The component R has just one limit point Q in C.§ Hence Q must be a cut-point of M. There is a simple closed curve containing Q and lying in C. Then Q is a cut-point of M which lies on a simple closed curve of M.

Theorems 4 and 6 to 13 inclusive of my paper, "On the Structure of a Plane Continuous Curve",¶ hold in our more general space with the same proofs as given for two dimensions as these proofs depend only on theorems 10, 11, 13, 14 and 15 of the present paper and theorems 7 and 10, part (2) of my paper, "Concerning the Arc-Curves and Basic Sets of a Continuous Curve",‖ all of which hold in our more general space.

NATIONAL RESEARCH FELLOW IN MATHEMATICS,
THE UNIVERSITY OF TEXAS.

* For the two-dimensional case of this theorem, see reference (H), Theorem 5.

† This very simple proof of this theorem was suggested by G. T. Whyburn. It is somewhat shorter than my original proof.

‡ See reference (G), Theorem 3, and a statement following Theorem 10 of the present paper on the validity of this result for the more general space considered here. A cyclicly connected continuous curve C which is a subset of a continuous curve M is said to be a *maximal cyclic curve* of M if C is not a proper subset of any cyclicly connected continuous curve which is a subset of M.

§ See reference (G), Theorem 2, and a statement on this theorem in the present space following Theorem 10 of the present paper.

¶ See reference (H).

‖ See reference (I) and a statement following Theorem 12 of the present paper on the validity of these theorems in our more general space.

On A Certain Function of the Masses in the Problem of Three Bodies.

By H. E. Buchanan.

Introduction. In his discussion of the stability of an infinitesimal body near the equilateral triangle positions Professor Moulton * found a certain function of the two finite masses whose sign determined the character of the motion of the infinitesimal body. He chose the two finite masses to be μ and $1-\mu$ and found the infinitesimal body to be in stable equilibrium if

$$1-27\mu(1-\mu)\geqq 0,$$

and this inequality is satisfied by $\mu \leqq .0385$.

It is the purpose of this paper to discuss the corresponding function when all the masses are finite.

The foundation for the discussion has already been laid in the author's recent paper on Periodic Orbits † near the Equilateral Triangle Solutions. We make use of the equations (5), (6) and (7) of that paper, together with the results obtained from them. The function which we wish to discuss is $(m_1+m_2+m_3)^2-27(m_1 m_2+m_1 m_3+m_2 m_3)=f(m_1, m_2, m_3)$. If $f(m_1, m_2, m_3)>0$, the characteristic exponents are pure imaginaries, in pairs, except for the double root zero. If $f(m_1, m_2, m_3)=0$ there appears another pair of double roots, $+i\omega/2^{1/2}$ and $-i\omega/2^{1/2}$. If $f(m_1, m_2, m_3)<0$ there appears, in addition to the double root zero, a pair of conjugate complex roots. It follows that the equilateral triangle solutions are always unstable for every arrangement of the masses. We designate the three cases above:

Case I. $f(m_1, m_2, m_3)>0$.
Case II. $f(m_1, m_2, m_3)=0$.
Case III. $f(m_1, m_2, m_3)<0$.

The cone $f(m_1, m_2, m_3)=0$. Since $f(m_1, m_2, m_3)$ is homogeneous and of the second degree in m_1, m_2, and m_3, $f(m_1, m_2, m_3)=0$ may be regarded as the equation of a cone, m_1, m_2, m_3 being the rectangular coordinates of any point. This cone has its vertex at the origin and, from the symmetry, the line $m_1=m_2=m_3$ is its axis. We wish to define precisely those regions in which

* Moulton, *Celestial Mechanics*, revised ed., p. 307.
† *American Journal of Mathematics*, Vol. 50, No. 4 (1928).

all the masses are positive and $f(m_1, m_2, m_3) \gtreqless 0$. Further, if one of the masses is regarded as given we shall find the limiting values of the other two in each of the three cases. Finally, if $m_3 = 0$, we shall get Moulton's results and obtain a geometrical interpretation for them.

Rotating the $m_1\, m_3$ plane clockwise through 45° changes the equation of the cone to

$$27x^2 + 2m_2^2 - 23z^2 - 50\,(2)^{1/2}\, m_2 z = 0.$$

Rotating the $m_2 z$ plane counter clockwise through the proper angle to remove the $m_2 z$ term we find

$$3x^2 + 3y^2 - (16/3)z^2 = 0,$$

which is a circular cone. The generating angle is approximately 53° 7′.

The Traces on the reference planes. Returning to the original variables we find the traces on the $m_1\, m_2$ plane by making $m_3 = 0$. This gives

$$m_1^2 + m_2^2 - 25 m_1 m_2 = 0,$$

or

$$m_1/m_2 = .04005, \quad m_1/m_2 = 24.9599.$$

These represent two straight lines whose inclinations are, approximately 2° 17′ and 87° 43′. Since m_1, m_2 and m_3 are interchangeable it follows that the traces on the other two reference planes are similarly situated.

Let us agree that $m_3 \geqq m_2 \geqq m_1$ and let us cut the cone by the plane

$$m_1 + m_2 + m_3 = C$$

where c is the largest value m_3 may have. Since the masses are always positive we confine our attention to that portion of space bounded by the coordinate planes and the plane

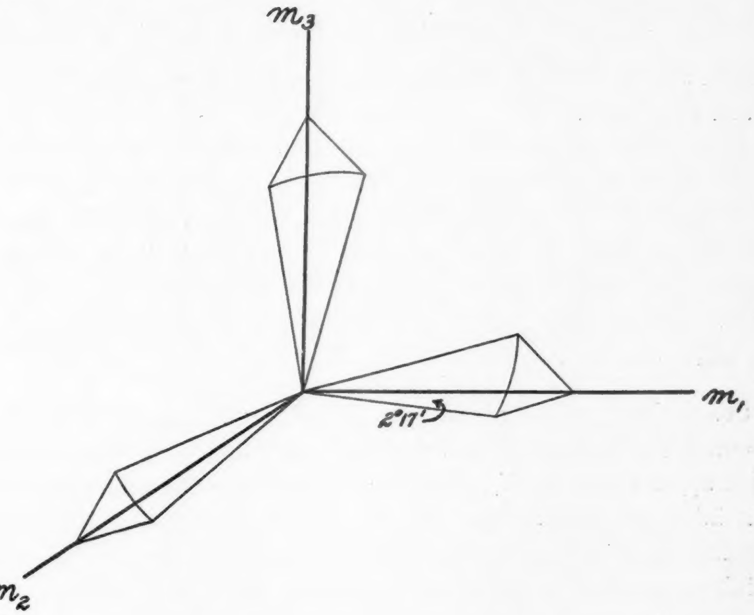
$$m_1 + m_2 + m_3 = C.$$

It is evident that $f(m_1, m_2, m_3) < 0$ for $m_1 = m_2 = m_3$, that is, for any point on the axis of the cone. The function can change sign only as (m_1, m_2, m_3) passes through the surface of the cone. Hence the only values of the masses which give rise to case I are determined by points inside the slender, rapier-like volumes extending out each axis, bounded by the coordinate planes and the cone. These are shown in the adjoining figure. For the purposes of drawing we were obliged to make them more blunt than they are in fact. If m_3 may be regarded as given then the largest value m_2 may have is $.04005 m_3$ and for these values $m_1 = 0$. But if $m_2 < .04005 m_3$ then m_1 may have any value from zero up to

$$\tfrac{1}{2}\left[25(m_2+m_3)-\{621(m_2^2+m_3^2)+1350\,m_2m_3\}^{1/2}\right],$$

which in turn must be less than $.04005m_3$.

The Infinitesimal Case. If $m_3=0$ and if we take the sum of the masses as unity we may find the limiting values of m_1 and m_2 by solving $m_1+m_2=1$ simultaneously with $m_1=.04005\,m_2$. The result is $m_1=.0385$, $m_2=.9615$. If we had solved simultaneously with the line near the m_2 axis the values of m_1 and m_2 would have been interchanged. This result checks exactly with Moulton's.



An Application. The mass of Jupiter is $954.8/10^6$ of the mass of the sun. If we let $m_3=1$ represent the sun, and $m_2=.0009548$ represent Jupiter, then the largest value m_1 can have, in order that Case I may hold, is $m_1=.039$. For these values, and for all smaller positive values of m_1 the equilateral triangle position is stable except for the presence, in the X_i and Y_i equations, of terms which are linear in t. A similar computation made for the earth and sun shows that Case I will hold for a body at the equilateral triangle position 13,000 times the mass of the earth.

If Gylden * and Moulton † are correct in their explanation of the Gegen-

* *Bulletin Astronomique*, Vol. I.

† Moulton, *Astronomical Journal*, N. 483.

schein, then it seems that there ought to be a larger accumulation near the equilateral triangle positions than near the straight line positions, for the latter give rise to terms of the type e^{at} in addition to linear terms in t.* This accumulation, however, would not be as readily visible as the Gegenschein because of its greater distance from the earth and because less than half the surfaces of any bodies there would appear illuminated, as seen from the earth.

Cases II and III. If m_1, m_2 and m_3 are such that $f(m_1, m_2, m_3) = 0$, that is for any point on the cone, the characteristic exponents are

$$\pm i\omega, \ + i\omega/2^{1/2}, \ + i\omega/2^{1/2}, \ - i\omega/2^{1/2}, \ - i\omega/2^{1/2}, \ 0, \ 0.$$

Hence in this case also the positions are unstable, due to the presence of terms $K_4 t$ and $K_6 t e^{i\omega t/2^{1/2}}$.

Finally, for any point inside the cone the positions are unstable, due to the presence of terms of the type $K_4 t$ and of the type $K_6 e^{at} e^{i\beta t}$.

Thus in any system consisting of a sun and a planet, if the mass of the planet is less than 4% of its principal, Case I holds for any mass at the triangle point which is less than 1/10 of 1% of the principal. The planets of our system all come under Case I but the binary star systems undoubtedly come under Case III.

Conclusion. If we assume the initial conditions, $X_i = \alpha_i$, $Y_i = \beta_i$, $X' = \gamma_i$, $Y'_i = \delta_i$, $i = 1, 3$ at $t = 0$ and apply these conditions to equations (7)† it appears that the $K_{ij} (j = 1 \cdots 8)$ are linear homogeneous functions of the initial constants. Therefore K_{14} may be made zero if the initial constants are chosen so that a linear relation is satisfied. In this event seven of the initial constants are completely arbitrary and small oscillations of the three finite bodies will contain only periodic terms in Case I.

If the masses are such that Case II holds then three linear relations must be satisfied by the initial constants in order that only periodic terms may be present in equations (7). If Case III holds then five linear relations must hold between the initial constants in order that only periodic terms appear in the solutions.

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* Buchanan, "Periodic Orbits," *American Journal of Mathematics*, Vol. 45, No. 2, (1923).

† *American Journal of Mathematics*, Vol. 50, Oct. (1928).

Invariants of Sets of Points under Inversion.

By Joseph William Peters.

1. *Introduction.* Two points, P and P', on a line with a point O are called inverse points if $OP \cdot OP' = \kappa$, where κ is a real constant. The point O is the center of the inversion. The transformation applies not only to points in a plane but to points in flat spaces of any number dimensions. For $\kappa > 0$, there is in the plane a circle of fixed points, in three dimensions a sphere of fixed points, and so on. To every point P different from O, there corresponds a unique finite point. In order to have a one-to-one correspondence in all cases, we close our space by a single point at infinity which is the inverse of the center, O. As a consequence of this assumption, circles and lines in the plane are inversely equivalent; spheres and planes in three dimensions are inversely equivalent; and so on. A further important property of inversions concerns the relationship between the distance from P to Q and the distance between the inverse points P' and Q'. If O is the center of inversion and κ the constant of inversion, the relation is

$$PQ = (\kappa / OP' \cdot OQ') \cdot P'Q'.$$

With these general remarks concerning inversions in mind, let us now proceed to the discussion of invariants under inversion.

2. *Inversive Properties of Sets of Points.* We shall first take the case of three points. Let (ij) represent the distance between the points i and j and let $(ij) = (ji)$. If 0 be a variable point in the plane of the three fixed points $1, 2, 3$, consider the equations

$$(01)(23) = (02)(31) = (03)(12). \tag{2.1}$$

Under an inversion with any other point in the plane of $1, 2, 3$ as a center, these equations are reproduced in the primed numbers. Hence we will say that they are invariant under inversions, or simply inversive.

The equations (2.1) represent three circles which intersect in two real points. For if we let the point 3 approach infinity, the equations of the three circles become

$$(01) = (12) \qquad (02) = (12) \qquad (01) = (02).$$

The first circle has its center at 1 and radius (12), the second has its center

at 2 and radius (12), hence they intersect on the perpendicular bisector of (12) which is the third circle.

Since the circles (2.1) intersect in two real points, let us move one of these points to infinity by using it as the center of an inversion. The circles (2.1) now become straight lines with the equations

$$(01) = (02) = (03),$$

and the triangle 1, 2, 3 becomes equilateral since

$$(12) = (23) = (31).$$

It is now evident that each circle passes through one vertex of the triangle 1, 2, 3 and has the other two vertices as inverse points. Hence the equations $(01) = (02) = (03)$ represent the Apollonian Circles of the equilateral triangle 1, 2, 3.

In general, the equations (2.1) represent the Apollonian Circles of the triangle 1, 2, 3. Their two points of intersection are called the Hessian points of the triangle.

Consider now the equations

$$(01)(23) \pm (02)(13) \pm (03)(12) = 0 \tag{2.2}$$

using all combinations of signs. The equations are inversive. If we invert with one of the Hessian points as a center, the triangle 1, 2, 3 becomes equilateral as before, and the equations (2.2) become

$$(01) \pm (02) \pm (03) = 0.$$

The case where the signs are all positive is an inversion in the circumcircle of the triangle 1, 2, 3.

$(01) + (02) - (03) = 0$ represents the arc of the circumcircle from the point 1 to the point 2.

$(01) - (02) + (03) = 0$ represents the arc of the circumcircle from the point 3 to the point 1.

$(01) - (02) - (03) = 0$ represents the arc of the circumcircle from the point 2 to the point 3.

The point simultaneously satisfying $(01) = (02) + (03)$ and $(02) = (03)$ is on the circumcircle and is called the counterpoint of 1. Similarly $(02) = (01) + (03)$ and $(01) = (03)$ give the counterpoint of 2 and $(03) = (01) + (02)$ and $(01) = (02)$ give the counterpoint of 3.

To summarize, we may say the invariant relations of three points under inversion give the Hessian points; the Apollonian Circles; the circumcircle of the three points, not as a whole but in three arcs which make it complete;

an inversion in the circumcircle; and lastly the counterpoints of the three given points.

Take now the case of four points. Let 1, 2, 3, 4 be four points in a three-dimensional space with 0 as a variable point in that space and consider the equations homogeneous in the points 0, 1, 2, 3, 4

$$(2.3)\qquad \begin{aligned}&(01)^2(23)(34)(42) = (02)^2(34)(41)(13)\\ &= (03)^2(41)(12)(24) = (04)^2(12)(23)(31).\end{aligned}$$

These six equations are inversive. They represent six spheres which intersect in two points, for if the point 4 approaches infinity the equations (2.3) become

$$(01)^2(23) = (02)^2(13) = (03)^2(12) = (12)(23)(13).$$

The spheres $(01)^2 = (12)(13)$ and $(02)^2 = (12)(23)$ will intersect if

$$(01) + (02) > (12)$$

or

$$(13)^{1/2} + (23)^{1/2} > (12)^{1/2}.$$

We assume that none of the four points coincide, so $(13) + (23) > (12)$. This inequality implies $(13)^{1/2} + (23)^{1/2} > (12)^{1/2}$. A similar proof shows that $(01)^2 = (12)(13)$ and $(03)^2 = (23)(13)$ intersect and likewise $(02)^2 = (12)(13)$ and $(03)^2 = (23)(13)$ intersect. These three spheres will intersect in two points and from the original conditions imposed by the equations the other three spheres must also pass through these two points, called the canonizant pair.

Let us put one of these canonizant points at infinity by an inversion with that point as a center. Then the six spheres become six planes whose equations are given by

$$(01)^2 = (02)^2 = (03)^2 = (04)^2.$$

The distances between the four points will then satisfy the following equations:

$$(23)(34)(42) = (34)(41)(13) = (14)(12)(24) = (12)(23)(13),$$

which reduce to

$$(12)^2 = (34)^2, \quad (13)^2 = (24)^2, \quad (14)^2 = (23)^2.$$

Four points in a three-dimensional space are always on a sphere, and these equations arrange the four points so that, if the center of the sphere is the origin, the points may be given coordinates (a, b, c), $(a, -b, -c)$, $(-a, b, -c)$, $(-a, -b, c)$. In this same reference frame, the Cartesian equations of the six planes are

$$\begin{array}{ll}
(01)^2=(02)^2 & by+cz=0, \\
(01)^2=(03)^2 & ax+cz=0, \\
(01)^2=(04)^2 & ax+by=0, \\
(02)^2=(03)^2 & ax-by=0, \\
(02)^2=(04)^2 & ax-cz=0, \\
(03)^2=(04)^2 & by-cz=0.
\end{array}$$

Thus when one of the canonizant points is at infinity, the other is at the center of the sphere on the four points.

Consider now the following equations, using all combinations of signs:

$$(2.4) \qquad (01)^2(23)(34)(42) \pm (02)^2(34)(41)(13) \pm (03)^2(41)(12)(24) \pm (04)^2(12)(23)(31)=0.$$

The eight equations are inversive and represent spheres or inversions. To study these equations, it will be convenient to put one of the canonizant points at infinity, as was done before. Then the equations (2.4) become

$$(01)^2 \pm (02)^2 \pm (03)^2 \pm (04)^2=0,$$

and the points $1, 2, 3, 4$ may be given coordinates (a, b, c), $(a, -b, -c)$, $(-a, b, -c)$, $(-a, -b, c)$, respectively. To determine the nature of these spheres or inversions, we will find their equations in the Cartesian coordinate system determined by the naming of the four points.

$$(01)^2+(02)^2+(03)^2+(04)^2=0$$

becomes
$$x^2+y^2+z^2=-(a^2+b^2+c^2).$$

This represents an inversion with $(0, 0, 0)$ as the center. The sphere of fixed points does not exist. Under this inversion

$$\begin{array}{lcl}
(\ \ a, \ \ b, \ \ c) & \rightarrow & (-a, -b, -c), \\
(\ \ a, -b, -c) & \rightarrow & (-a, \ \ b, \ \ c), \\
(-a, \ \ b, -c) & \rightarrow & (\ \ a, -b, \ \ c), \\
(-a, -b, \ \ c) & \rightarrow & (\ \ a, \ \ b, -c).
\end{array}$$

The right-hand column gives four points known as the counterpoints of the first set. The equation

$$(01)^2+(02)^2+(03)^2-(04)^2=0$$

becomes in the Cartesian reference frame

$$(x-a)^2+(y-b)^2+(z+c)^2=0.$$

It evidently represents the point $(a, b, -c)$. Likewise

$$(01)^2+(02)^2-(03)^2+(04)^2=0$$

represents the point $(a, -b, c)$,

$$(01)^2-(02)^2+(03)^2+(04)^2=0$$

represents the point $(-a, b, c)$,

and
$$(01)^2-(02)^2-(03)^2-(04)^2=0$$

represents the point $(-a, -b, -c)$. Thus for an odd number of minus signs in the equations (2.4) the counterpoints of the four given points appear. The relation of the four points to the four counterpoints is mutual. The

sphere
$$(01)^2+(02)^2-(03)^2-(04)^2=0$$

represents the coordinate plane $x=0$, in the Cartesian system chosen.

Likewise
$$(01)^2-(02)^2+(03)^2-(04)^2=0$$

represents the coordinate plane $y=0$, and

$$(01)^2-(02)^2-(03)^2+(04)^2=0$$

represents the coordinate plane $z=0$. These three spheres (or planes) are mutually perpendicular and intersect on the canonizant points, which in this case are $(0, 0, 0)$ and ∞. The sphere containing the four points $1, 2, 3, 4$ and the planes $x=0$ and $y=0$ intersect in the points $(0, 0, \pm 1)$ if the sphere is regarded as being of unit radius. Likewise the sphere and the planes $x=0$ and $z=0$ intersect at $(0, \pm 1, 0)$, and the sphere and the planes $y=0$ and $z=0$ intersect at $(\pm 1, 0, 0)$. These six points are called the Jacobian points of $1, 2, 3, 4$ and the three spheres (or planes) $x=0$, $y=0$, $z=0$ are called the Jacobian spheres.

The inversive theory of four points gives, in short, six spheres which intersect in two points called the canonizant pair. It also gives the four counterpoints and an inversion connecting them with the original four points. Likewise the three Jacobian spheres and the six Jacobian points appear.

Since four points in three-dimensional space are always on a sphere, we may consider the special case when the sphere is a plane. The counterpoints will also be in the plane and the inversion sending them into the original

four points will be determined. Since the three Jacobian spheres intersect the sphere containing the four points, they will intersect the plane in three circles called the Jacobian circles. These circles will be perpendicular to each other and intersect in the six Jacobian points.

Let us now attempt to extend this theory to five points in a four-dimensional space. We will first consider the equations

$$\begin{aligned}\frac{(01)^3}{(12)(13)(14)(15)} &= \frac{(02)^3}{(12)(23)(24)(25)} = \frac{(03)^3}{(13)(23)(34)(35)} \\ &= \frac{(04)^3}{(14)(24)(34)(45)} = \frac{(05)^3}{(15)(25)(35)(45)}\,. \end{aligned} \tag{2.5}$$

These equations are inversive and represent sixteen hyperspheres. However, if we try to prove that the spheres intersect as in the case of four points, we find that they may not. Hence, if they do not intersect, we do not have a canonizant pair of points as we had in the other cases. We have, however, a canonizant circle.

Let us now consider the equations

$$\begin{aligned}\frac{(01)^3}{(12)(13)(14)(15)} &\pm \frac{(02)^3}{(12)(23)(24)(25)} \pm \frac{(03)^3}{(13)(23)(34)(35)} \\ &\pm \frac{(04)^3}{(14)(24)(34)(45)} \pm \frac{(05)^3}{(15)(25)(35)(45)} = 0. \end{aligned} \tag{2.6}$$

We have here sixteen equations representing hyperspheres or inversions. Let us suppose the five points so chosen that the equations (2.6) become

$$(01)^3 \pm (02)^3 \pm (03)^3 \pm (04)^3 \pm (05)^3 = 0.$$

The conditions imposed on the five points in that case are

$$\begin{aligned}(12)(13)(14)(15) &= (12)(23)(24)(25) = (13)(23)(34)(35) \\ &= (14)(24)(34)(45) = (15)(25)(35)(45).\end{aligned}$$

They do not seem to place the points in any symmetrical arrangement.

If we calculate the "power" or bilinear invariant of each hypersphere of the set

$$(01)^3 \pm (02)^3 \pm (03)^3 \pm (04)^3 \pm (05)^3 = 0$$

with itself, we find there are six inversions and ten real hyperspheres. The six inversions are given by the one equation with all plus signs and the five equations with one minus sign. The ten hyperspheres are given by the ten

equations with two minus signs. The geometric connection between these hyperspheres and inversions and the five points is not evident, and in this paper we will leave the theory here.

3. *The Cayley Determinants.* Cayley * has shown that for four points to be on a circle, the following determinant is zero,

$$C_4 \equiv \begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & 0 & \lambda_{23} & \lambda_{24} \\ \lambda_{31} & \lambda_{32} & 0 & \lambda_{34} \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & 0 \end{vmatrix} = 0$$

where λ_{ij} is the square of the distance between the points i and j and $\lambda_{ij} = \lambda_{ji}$. Five points on a sphere satisfy a similar relation, $C_5 = 0$, where C_5 is a five-rowed determinant similar to C_4.

In the same paper, Cayley gave the condition for three points to be on a line, namely

$$D_3 \equiv \begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & 1 \\ \lambda_{21} & 0 & \lambda_{23} & 1 \\ \lambda_{31} & \lambda_{32} & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

Four points in a plane satisfy the relation $D_4 = 0$ where D_4 is the determinant C_4 with a bordering row and column of units. Similarly five points in three-dimensional space satisfy $D_5 = 0$ where D_5 is the determinant C_5 with a bordering row and column of units.

The relations $C_4 = 0$ and $C_5 = 0$ are inversive, for if we replace λ_{ij} by $\kappa^2\lambda'_{ij}/\lambda'_{ci}\lambda'_{cj}$, where c is the center of the inversion and not one of the points, $C_4 = 0$ is reproduced in the primed letters, and likewise $C_5 = 0$ becomes $C_5' = 0$. If the center of the inversion, c, is one of the given points, then $C_4 = 0$ becomes $D_3 = 0$, the fourth point having gone into the point at infinity. Likewise if we invert with the point 5 as a center, $C_5 = 0$ becomes $D_4 = 0$. The relations $D_3 = 0$, $D_4 = 0$, $D_5 = 0$ are not inversive. If we invert with a point c, not in the line of 1, 2, 3 as a center, the point c goes to infinity, the point at infinity becomes a point 4, and $D_3 = 0$ becomes $C_4 = 0$. Likewise we may derive the condition that five points be on a sphere, $C_5 = 0$, by an inversion applied to $D_4 = 0$, where the center of the inversion must not be in the plane of the four points satisfying $D_4 = 0$.

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* Cayley, *Collected Mathematical Papers*, Vol. I, p. 1; also Salmon, *Conic Sections*, 6th ed. (1879), p. 134.

It is evident that determinants such as these may be set down for any number of points. $D_n = 0$ is the relation connecting any n points in S_{n-2}. $C_n = 0$ is the condition that n points be on a "round" or hypersphere in S_{n-2}. The relations $C_n = 0$ are all inversive.

The question arises as to what geometric interpretation we may give these determinants when they are not zero. The following two theorems will answer the question for three points and then extensions will be made to any number of points.

THEOREM 1. *If Δ be the area of the triangle formed by any three points, then*

$$D_3 = -16\Delta^2.$$

Proof: Three points on a line satisfy the relation $D_3 = 0$. If the point 3 is at a distance p from the line joining 1 and 2, then

$$\begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} - p^2 & 1 \\ \lambda_{21} & 0 & \lambda_{23} - p^2 & 1 \\ \lambda_{31} - p^2 & \lambda_{32} - p^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

On splitting this determinant up into sums, we find that $D_3 + 2p^2 D_2 = 0$. But $D_2 = 2\lambda_{12}$ and since $p^2\lambda_{12}$ is four times the square of the area of the triangle 1, 2, 3, then $D_3 = -16\Delta^2$.

THEOREM 2. *If r be the radius of the circumcircle of any three points not in a straight line, then $C_3 = -2r^2 D_3$.*

Proof: For three points not in a straight line, we have

$$\begin{vmatrix} 1 & \cos(12) & \cos(13) \\ \cos(21) & 1 & \cos(23) \\ \cos(31) & \cos(32) & 1 \end{vmatrix} = 0$$

where $\cos(ij)$ is the cosine of the angle subtended at the circumcenter by the arc of the circumcircle joining the points i and j and $\cos(ij) = \cos(ji)$. If we replace $\cos(ij)$ by $1 - \lambda_{ij}/2r^2$, where r is the radius of the circumcircle and λ_{ij} is the square of the distance between the points i and j, the determinant becomes

$$\begin{vmatrix} 1 & 1 - \lambda_{12}/2r^2 & 1 - \lambda_{13}/2r^2 \\ 1 - \lambda_{21}/2r^2 & 1 & 1 - \lambda_{23}/2r^2 \\ 1 - \lambda_{31}/2r^2 & 1 - \lambda_{32}/2r^2 & 1 \end{vmatrix} = 0.$$

After splitting this determinant up into a sum of determinants, we find $C_3 + 2r^2D_3 = 0$. Since from theorem 1, $D_3 = -16\Delta^2$, we have $C_3 = 32r^2\Delta^2$.

The following are the corresponding theorems for four points not in a plane, with a sketch of the proofs:

THEOREM 1′: *If V be the volume of the tetrahedron formed by the four points* 1, 2, 3, 4, then $D_4 = 288V^2$.*

Proof: If p be the distance from the plane of the points 1, 2, 3 to the point 4, we may replace in the relation $D_4 = 0$, λ_{i4} by $\lambda_{i4} - p^2$ $(i = 1, 2, 3)$. Splitting the resulting determinant up into sums, we find $D_4 + 2p^2D_3 = 0$.

Since $$D_3 = -16\Delta^2, \quad D_4 = 32p^2\Delta^2.$$

But since $p\ \Delta = 3V$, we have $D_4 = 288V^2$.

THEOREM 2′: *If r be the radius of the sphere on the four points, then*

$$C_4 = -2r^2D_4.$$

Proof: Four points in a three-dimensional space always satisfy the relation †

$$\begin{vmatrix} 1 & \cos(12) & \cos(13) & \cos(14) \\ \cos(21) & 1 & \cos(23) & \cos(24) \\ \cos(31) & \cos(32) & 1 & \cos(34) \\ \cos(41) & \cos(42) & \cos(43) & 1 \end{vmatrix} = 0,$$

where (ij) is the arc of the great circle joining the points i and j on the sphere on the four points. Replacing $\cos(ij)$ in this determinant by $1 - \lambda_{ij}/2r^2$ and splitting up the resulting determinant into sums, we find

$$C_4 + 2r^2D_4 = 0.$$

Since $D_4 = 288V^2$, we have

$$C_4 = -576r^2V^2.$$

These theorems can be extended to any number of points. Thus we have

THEOREM I. *If V be the content of the solid formed by n points in S_{n-1}, then $D_n = kV^2$, where $k = (-1)^n \cdot (2)^{n-1}\ [(n-1)\,!]^2$.*

* Salmon, *Analytic Geometry of Three Dimensions*, 3d ed. (1874), p. 34.

† Salmon, *loc. cit.*, p. 35.

THEOREM II. *If r be the radius of the hypersphere in S_{n-1} on which the n points lie, then $C_n = -2r^2D_n$, or $C_n = 2kr^2V^2$, where k is defined as above.*

4. *The Doubly Bordered Cayley Determinant.* Four points in a plane satisfy the relation $D_4 = 0$. If about the point 4, a circle of radius ρ is drawn, then

$$\lambda_{14} = (\delta_1 + \rho)^2, \quad \lambda_{24} = (\delta_2 + \rho)^2, \quad \lambda_{34} = (\delta_3 + \rho)^2,$$

where $\delta_1, \delta_2, \delta_3$ represent perpendicular distances from the points $1, 2, 3$, respectively, to the circumference of the circle. The relation, a quadratic in ρ, is now written

$$\begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & (\delta_1 + \rho)^2 & 1 \\ \lambda_{21} & 0 & \lambda_{23} & (\delta_2 + \rho)^2 & 1 \\ \lambda_{31} & \lambda_{32} & 0 & (\delta_3 + \rho)^2 & 1 \\ (\delta_1 + \rho)^2 & (\delta_2 + \rho)^2 & (\delta_3 + \rho)^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

If now the point 4 moves off to infinity, ρ becomes infinite, the circle becomes a straight line, and the coefficient of ρ^2 in the above relation is zero. Therefore

$$2 \begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & \delta_1 & 1 \\ \lambda_{21} & 0 & \lambda_{23} & \delta_2 & 1 \\ \lambda_{31} & \lambda_{32} & 0 & \delta_3 & 1 \\ \delta_1 & \delta_2 & \delta_3 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & 1 \\ \lambda_{21} & 0 & \lambda_{23} & 1 \\ \lambda_{31} & \lambda_{32} & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

The determinant on the left is Cayley's C_3 doubly bordered, first with δ's, then with units. The determinant on the right is D_3 which is equal to $-16\Delta^2$ where Δ is the area of the triangle formed by the three points $1, 2, 3$. Let $\lambda_{23} = a^2$, $\lambda_{13} = b^2$, $\lambda_{12} = c^2$ and let A, B, C be the interior angles of the triangle $1, 2, 3$ at the points $1, 2, 3$ respectively. Then if the left-hand determinant is expanded, we have

$$(4.1) \qquad a^2\delta_1^2 + b^2\delta_2^2 + c^2\delta_3^2 - 2ab \cos C\, \delta_1\delta_2 - 2ac \cos B\, \delta_1\delta_3 - 2bc \cos A\, \delta_2\delta_3 = 4\Delta^2.$$

The $\delta_1, \delta_2, \delta_3$ now represent the perpendicular distances from the points $1, 2, 3$ to a straight line. For a given δ_2 and δ_3 there are ordinarily two straight lines satisfying the equation. They are common tangents to the

circles with centers at 2 and 3 and radii δ_2 and δ_3 respectively. They are internal or external pairs according as 2 and 3 are on the opposite or the same side of the lines. The internal tangents coincide when $\delta_2 - \delta_3 = \pm a$ and likewise the external tangents coincide when $\delta_2 - \delta_3 = \pm a$. Algebraically we may therefore say that the discriminant of equation (4.1) with respect to δ_1, when set equal to zero, must reduce to $\delta_2 - \delta_3 = \pm a$. The discriminant set equal to zero gives

$$b^2 \sin^2 C\, \delta_2{}^2 + c^2 \sin^2 B\, \delta_3{}^2 - 2bc\,(\cos A + \cos B \cos c)\, \delta_2\delta_3 = 4\Delta^2.$$

This equation should reduce to $(\delta_2 - \delta_3)^2 = a^2$. Therefore the following relations must be satisfied.

$$b^2 \sin^2 C = bc\,(\cos A + \cos B \cos C),$$
$$b/c = \sin B / \sin C, \quad ab \sin C = 2\Delta.$$

The first relation is a form of the law of cosines for a triangle. The second is the law of sines. The third gives an expression for the area. The first relation gives

$$b^2 \sin^2 C = bc \cos A + bc \cos(B + C) + bc \sin B \sin C,$$

which, upon using the second relation becomes $\cos A = -\cos(B + C)$, whence $A + B + C = \pi$. Here we have sufficient information about the triangle to determine any of its properties. Can we do the same for four points in S_3?

To find the properties of the tetrahedron, or four point in S_3, start with the five points satisfying the relation $D_5 = 0$. About the point 5 draw a sphere of radius ρ, and let $\delta_1, \delta_2, \delta_3, \delta_4$ be perpendicular distances from the points $1, 2, 3, 4$ to the sphere. Then in the relation $D_5 = 0$, set $\lambda_{i5} = (\delta_i + \rho)^2$, where $i = 1, 2, 3, 4$. If now the point 5 moves off to infinity in a given direction, ρ becomes infinite, the sphere becomes a plane, and the coefficient of ρ^2 in the determinant $D_5 = 0$ is zero. Therefore

$$\begin{vmatrix} 0 & \lambda_{12} & \lambda_{15} & \lambda_{14} & \delta_1 & 1 \\ \lambda_{21} & 0 & \lambda_{23} & \lambda_{24} & \delta_2 & 1 \\ \lambda_{31} & \lambda_{32} & 0 & \lambda_{34} & \delta_3 & 1 \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & 0 & \delta_4 & 1 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{vmatrix} = 144V^2,$$

where V is the volume of the tetrahedron 1, 2, 3, 4. The above determinant is C_4, doubly bordered, first with δ's, then with units. Remembering that $D_3 = -16\Delta^2$, we have on expanding the above determinant

$$
\begin{aligned}
(4.2)\qquad & \Delta_1^2\delta_1^2 + \Delta_2^2\delta_2^2 + \Delta_3^2\delta_3^2 + \Delta_4^2\delta_4^2 \\
& - 2\Delta_1\Delta_2\delta_1\delta_2 \cos(12) - 2\Delta_1\Delta_3\delta_1\delta_3 \cos(13) \\
& - 2\Delta_1\Delta_4\delta_1\delta_4 \cos(14) - 2\Delta_2\Delta_3\delta_2\delta_3 \cos(23) \\
& - 2\Delta_2\Delta_4\delta_2\delta_4 \cos(24) - 2\Delta_3\Delta_4\delta_3\delta_4 \cos(34) = 9V^2,
\end{aligned}
$$

where $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ are the areas of the faces opposite the points 1, 2, 3, 4 respectively, and cos (12) is the cosine of the angle between the faces Δ_1 and Δ_2.

NOTE: In the expansion of the above determinant, we used the fact that

$$
\begin{vmatrix}
\lambda_{12} & \lambda_{23} & \lambda_{24} & 1 \\
\lambda_{13} & 0 & \lambda_{34} & 1 \\
\lambda_{14} & \lambda_{43} & 0 & 1 \\
1 & 1 & 1 & 0
\end{vmatrix} = -16\Delta_1\Delta_2 \cos(12).
$$

To prove this we choose a rectangular reference frame in the plane of the points 2, 3, 4, giving the points coordinates (x_2, y_2), (x_3, y_3), (x_4, y_4) respectively. The projection of the point 1 on this plane is given coordinates (x_1', y_1'). Then

$$
\begin{vmatrix}
x_2^2 + y_2^2 & -2x_2 & -2y_2 & 1 \\
x_3^2 + y_3^2 & -2x_3 & -2y_3 & 1 \\
x_4^2 + y_4^2 & -2x_4 & -2y_4 & 1 \\
1 & 0 & 0 & 0
\end{vmatrix}
\begin{vmatrix}
1 & x_1' & y_1' & x_1'^2 + y_1'^2 \\
1 & x_3 & y_3 & x_3^2 + y_3^2 \\
1 & x_4 & y_4 & x_4^2 + y_4^2 \\
0 & 0 & 0 & 1
\end{vmatrix} =
$$

$$
-16\,\Delta_1\Delta_2 \cos(12).
$$

$$
\text{Whence}\quad
\begin{vmatrix}
\lambda_{1'2} & \lambda_{23} & \lambda_{24} & 1 \\
\lambda_{1'3} & 0 & \lambda_{34} & 1 \\
\lambda_{1'4} & \lambda_{43} & 0 & 1 \\
1 & 1 & 1 & 0
\end{vmatrix} = -16\Delta_1\Delta_2 \cos(12).
$$

Replacing in this determinant $\lambda_{1'i}$ by $\lambda_{1i} - p^2$, $(i = 2, 3, 4)$, where p is the distance from the point 1 to the plane of 2, 3, 4, and then splitting the resulting determinant into sums of determinants, we have the result.

For given $\delta_2, \delta_3, \delta_4$ there are two values of δ_1, or two planes, satisfying the equation (4.2). When will these planes coincide? Take three balls, no two of the same size, and place them on a table. Their centers will represent the points 2, 3, 4 and their radii $\delta_2, \delta_3, \delta_4$. The plane of the table will be one tangent plane and there will be also an upper external tangent plane, for which a piece of cardboard may be used. If we now bring the three points of contact of the spheres with the table into a line, the upper tangent plane will swing around and finally coincide with the table. Hence the three points

2, 3, 4 will be in a plane through the line of contact and perpendicular to the table. Then

$$(4.3) \qquad a^2\delta_2^2 + b^2\delta_3^2 + c^2\delta_4^2 - 2ab\cos C\,\delta_2\delta_3 - 2ac\cos B\,\delta_2\delta_4 - 2bc\cos A\,\delta_3\delta_4 = 4\Delta_1^2,$$

where a, b, c are the sides of the face Δ_1 opposite the points 2, 3, 4 respectively, and A, B, C are the interior angles at the points 2, 3, 4.

The same relation is true when the internal tangent planes coincide. Hence the discriminant of equation (4.2) with respect to δ_1 must reduce to equation (4.3) when the tangent planes coincide. The discriminant when set equal to zero, gives

$$\begin{aligned}
&\Delta_2^2\sin^2(12)\,\delta_2^2 + \Delta_3^2\sin^2(13)\,\delta_3^2 + \Delta_4^2\sin^2(14)\,\delta_4^2\\
&- 2\Delta_2\Delta_3\delta_2\delta_3[\cos(23) + \cos(12)\cos(13)]\\
&- 2\Delta_2\Delta_4\delta_2\delta_4[\cos(24) + \cos(12)\cos(14)]\\
&- 2\Delta_3\Delta_4\delta_3\delta_4[\cos(34) + \cos(13)\cos(14)] = 9V^2.
\end{aligned}$$

This equation must reduce to equation (4.3). Therefore the following relations must hold true:

$$\begin{gathered}
\Delta_2\sin(12)/a = \Delta_3\sin(13)/b = \Delta_4\sin(14)/c,\\
\sin(12)\sin(13)\cos C = \cos(23) + \cos(12)\cos(13),\\
\sin(12)\sin(14)\cos B = \cos(24) + \cos(12)\cos(14),\\
\sin(13)\sin(14)\cos A = \cos(34) + \cos(13)\cos(14),\\
2\Delta_1\Delta_2\sin(12) = 3aV.^*
\end{gathered}$$

Here we have information about the tetrahedron. The first set of equations gives the law of sines for the dihedral angles, the second set gives the law of cosines. The last equation is a formula for the volume.

This theory can be extended to any number of points. The doubly-bordered Cayley determinant for n points can always be set down. On expanding the determinant, we will always have a quadratic expression in the δ's, such that the discriminant with respect to δ_1, when set equal to zero, must reduce to the relation on the δ's for $(n-1)$ points. The algebraic conditions imposed by this fact give information about the n point.

THE JOHNS HOPKINS UNIVERSITY.

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* Casey, *Spherical Trigonometry*, p. 133.

Independent Event Histories.*

By ARTHUR H. COPELAND.

It has been proved that there exist N *independent numbers* † associated with a given probability p, N being arbitrarily large. The condition that a set of numbers be independent is expressed by a certain set of equations which must be satisfied by these numbers. Thus in a sense, independence is the exception and not the rule. However, another point of view is possible with respect to this question. It was proved that there exists a set of *admissible numbers* having the power of the continuum. We are led to inquire whether a set of independent numbers can have the power of the continuum, or whether such a set is necessarily finite or denumerable. We shall see that it is possible for a set of independent numbers to have the power of the continuum, and hence from this point of view we cannot say that independence is exceptional.‡

It is important to notice that a set of numbers can be independent when these numbers are not admissible. As admissible numbers are the ones which are chiefly of interest in the theory of probability, we shall restrict ourselves in this paper to the discussion of numbers which are both independent and admissible. We shall introduce the following definition:

DEFINITION: A set of numbers, $x_1, x_2, x_3, \cdots x_N$, is an admissibly independent set if and only if there exists a corresponding set of numbers, $p_1, p_2, p_3, \cdots p_N$, such that every sub-set of numbers, $x_{\nu_1}, x_{\nu_2}, x_{\nu_3}, \cdots x_{\nu_n}$, of the set $x_1, x_2, x_3, \cdots x_N$, satisfies the condition that $x_{\nu_1} \cdot x_{\nu_2} \cdot x_{\nu_3} \cdots x_{\nu_n}$ belongs to to $A(p_{\nu_1} p_{\nu_2} p_{\nu_3} \cdots p_{\nu_n})$.

Let us observe that it is possible for two numbers, x and y, to be independent and admissible when the product, $x \cdot y$ is not admissible. That is, x and y can be admissible and independent without being admissibly independent. For example, we can take the numbers, x and $(2/1)x$, where x belongs to $A(p)$. Then $(2/1)x$ belongs to $A(p)$.§ We have the equations

* This paper was presented to the Society September 7, 1928. It is based on a paper by the author entitled "Admissible Numbers in the Theory of Probability," *American Journal of Mathematics*, Vol. 50 (1928), pp. 535-552.

† The reader is referred, for notation and for the definition of independent and admissible numbers, to the memoir just cited, §§ I, III.

‡ See Theorems 2 and 3 below, also see definition of the independence of infinite sets, p. 613.

§ See the memoir cited, Theorem 13.

$$p\,[(1/3)\{x\cdot(2/1)x\}] = p\,[(1/3)x\cdot(2/3)x] = p^2 \text{ and}$$
$$p\,[(2/3)\{x\cdot(2/1)x\}] = p\,[(2/3)x\cdot(3/3)x] = p^2, \text{ but}$$
$$p\,[(1/3)\{x\cdot(2/1)x\}\cdot(2/3)\{x\cdot(2/1)x\}] =$$
$$p\,[(1/3)x\cdot(2/3)x\cdot(3/3)x] = p^3 \neq p^4.$$

Hence $x\cdot(2/1)x$ is not admissible. To establish the independence of x and $(2/1)x$ we have to prove that $p\,[x\cdot(2/1)x] = p^2$. We see that $p\,[(r/n)\{x\cdot(2/1)x\}] = p\,[(r/n)x\cdot(r+1/n)x] = p^2$, if $r = 1, 2, 3 \cdots, n-1$. Hence the independence of these numbers is a simple consequence of the following theorem.

THEOREM 1.* *If x is such that $p\{(r/n)x\} = p$ for every n and every r such that $0 < r \leqq n - q$, where q is a fixed integer, then $p(x) = p$.*

By hypothesis, if we have given a positive number, ϵ, then there exists a number, μ_0 such that

(a) $p - \epsilon/3 < p_\mu\{(r/n)x\} < p + \epsilon/3$ *if* $\mu \geqq \mu_0$ and $0 < r \leqq n - q$.

Let $\nu = \mu n + m$ where $0 \leqq m < n$, then

$$p_\nu(x) = \sum_{r=1}^{n-q} \mu\, p_\mu\{(r/n)x\}/\nu + \sum_{r=n-q+1}^{n} \mu\, p_\mu\{(r/n)x\}/\nu + \sum_{k=n\mu+1}^{n\mu+m} x^k/\nu.$$

But $0 \leqq p_\mu\{(r/n)x\} \leqq 1$ if $n - q < r \leqq n$, and $0 \leqq \sum_{k=n\mu+1}^{n\mu+m} x^k/\nu \leqq m/\nu$. Hence

$$(p - \epsilon/3)(n - q)\mu/\nu \leqq p_\nu(x) \leqq (p + \epsilon/3)(n - q)\mu/\nu + q\mu/\nu + m/\nu.$$

Since $(1 - q/n)\mu/(\mu + 1) \leqq (n - q)\,\mu/\nu \leqq 1$, we have the inequalities

$$p\mu/(\mu + 1) - \epsilon/3 - q/n \leqq p_\nu(x) \leqq p + \epsilon/3 + (q\mu + m)/(n\mu + m).$$

These inequalities can be replaced by the stronger inequalities

$$p - (\epsilon/3 + q/n + 1/\mu) \leqq p_\nu(x) \leqq p + (\epsilon/3 + q/n + 1/\mu).$$

Let us first choose n so that $q/n < \epsilon/3$, and next choose μ_0 so that condition (a) is satisfied and so that $1/\mu_0 < \epsilon/3$. Then $|\, p_\nu(x) - p\,| < \epsilon$ if $\nu \geqq n\mu_0$. Therefore $p(x) = p$.

We shall now consider infinite sets of admissible independent numbers, such sets being defined as follows:

DEFINITION. An infinite set, E, is an admissibly independent set if and only if every finite sub-set of E is admissibly independent.

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* Special cases of this theorem were used in proving Theorems 11 and 13 in the memoir cited.

THEOREM 2. *There exists an admissibly independent set, E, having the power of the continuum.*

In order to prove this theorem we shall make use of the following lemma:

LEMMA. For every number, a, such that $0 < a < 1$, there exists a sequence of integers, $q_s(a)$ $(s = 1, 2, 3, \cdots)$ such that $q_s(a) < s$ and such that if $a_1 \neq a_2$, then the equation, $q_s(a_1) = q_s(a_2)$, has only a finite number of solutions.*

Let $q_s(a)$ be such that $(q_s(a) + 1)/s > a \geqq q_s(a)/s$. Then $| q_s(a)/s - a | < 1/s$, and therefore $\lim_{s\to\infty} q_s(a)/s = a$. Let a_1 and a_2 be two numbers such that $0 < a_1, a_2 < 1$ and let us assume that $a_1 < a_2$. Let ϵ be such that $a_2 - \epsilon > a_1$. Then we can choose s_0 such that $a_2 - q_s(a_2)/s < \epsilon$ if $s \geqq s_0$. Then $q_s(a_2)/s > a_2 - \epsilon > a_1 > q_s(a_1)/s$ if $s \geqq s_0$, and hence the equation $q_s(a_1) = q_s(a_2)$, has at most s_0 solutions. This completes the proof of the lemma.

Corresponding to any number, a, such that $0 < a < 1$, we can define a number, $x(a)$ such that the digits $\nu_s + 1$ to ν_{s+1} of $x(a)$ are the same as the digits 1 to N_s of $(q_s(a)/s)y$, where $\nu_s = N_1 + N_2 + N_3 \cdots + N_{s-1}$ and where y belongs to $A(p)$. We wish to show that the integers, $N_1, N_2, N_3, \cdots N_s, \cdots$, can be so selected that the set of numbers, $x(a)$ is admissibly independent. We have to prove that if $a_1, a_2, a_3, \cdots a_\alpha$ is any set of distinct numbers such that $0 < a_1, a_2, a_3, \cdots a_\alpha < 1$, then $x(a_1) \cdot x(a_2) \cdots x(a_\alpha)$ belongs to $A(p^\alpha)$. Let

$$Y_{n,s} = (r_1/n)\{(\rho_1/s)y \cdot (\rho_2/s)y \cdots (\rho_\beta/s)y\} \cdot (r_2/n)\{(\rho_1/s)y \cdots (\rho_\beta/s)y\} \cdots (r_k/n)\{(\rho_1/s)y \cdots (\rho_\beta/s)y\}$$

where $n \leqq s$ and $r_1, r_2, r_3, \cdots r_k$ is any set of distinct positive integers such that $r_i \leqq n$, and where $\rho_1, \rho_2, \rho_3 \cdots \rho_\beta$ is any set of positive integers such that $\rho_j \leqq s$. Then if $\rho_1, \rho_2 \cdots \rho_\beta$ are distinct, $p(Y_{n,s}) = p^{k\beta}$.

Let $\epsilon_1 > \epsilon_2 > \cdots \epsilon_s > \cdots$ be such that $\lim_{s\to\infty} \epsilon_s = 0$. We can define two sets of integers, $M_1, M_2, M_3, \cdots M_s, \cdots$, and $N_1, N_2, N_3, \cdots N_s, \cdots$, such that

(a) $\quad | p_N(Y_{n,s+1}) - p^{k\beta} | < \epsilon_s/3$ if $N \geqq M_s/n$

(b) $\quad | p_N(Y_{n,s}) - p^{k\beta} | + (\nu_s + M_s)/N_s < \epsilon_s/3$ if $N \geqq N_s/n$

where $n \leqq s$ and where $\rho_1, \rho_2 \cdots \rho_\beta$ are distinct and $\rho_j \leqq s$. The numbers,

---

* It is interesting to notice that if the condition $q_s(a) < s$ were replaced by the condition $q_s(a) < N$, N being fixed, then the lemma would no longer be true.

N_s and M_s, can be chosen so that conditions (a) and (b) hold for all possible numbers, $Y_{n,s}$ and at the same time, the numbers, N_s, can be chosen so that ν_s/n is an integer if $n \leqq s$. Then N_s/n is an integer. Let

$$X_n = (r_1/n)\{x(a_1)\cdot x(a_2)\cdots x(a_\alpha)\}\cdot(r_2/n)\{x(a_1)\cdots x(a_\alpha)\}\\ \cdots(r_k/n)\{x(a_1)\cdots x(a_\alpha)\}.$$

Then digits $\nu_s/n+1$ to ν_{s+1}/n of X_n are the same as digits 1 to N_s/n of $Y_{n,s}$, if $\rho_j = q_s(a_j)$ and $\beta = \alpha$. In order that conditions (a) and (b) can be made to hold for this choice of the number, $Y_{n,s}$, the integers, $q_s(a_1)$, $q_s(a_2)$, $q_s(a_3), \cdots q_s(a_\alpha)$, must be distinct. However, it follows from the lemma that a number, s_0, can be chosen so that these integers are distinct if $s \geqq s_0$.

In the work which follows it will be assumed that $s \geqq s_0$ and $s \geqq n$. Then

(c) $\quad \left| p_N(X_n) - \dfrac{N_s}{nN}\, p_{(N_s/n)}(Y_{n,s}) \right| \leqq (nN - N_s)/nN < (\nu_s + M_s)/N_s < \epsilon_s/3$

if $\quad \nu_{s+1}/n \leqq N \leqq (\nu_{s+1}+M_s)/n.$

Combining (b) and (c) we get

(d) $$\left| p_N(X_n) - \frac{N_s}{nN}\, p^{k\alpha} \right| < (1 + N_s/nN)\epsilon_s/3 < 2\epsilon_s/3.$$

Combining (d) with the inequality

(e) $$\left| p^{k\alpha}\cdot N_s/nN - p^{k\alpha} \right| < 1 - N_s/nN < \epsilon_s/3$$

we get

(f) $$\left| p_N(X_n) - p^{k\alpha} \right| < \epsilon_s \text{ if } \nu_{s+1}/n \leqq N \leqq (\nu_{s+1}+M_s)/n.$$

If $(\nu_{s+1}+M_s)/n \leqq N \leqq \nu_{s+2}/n$, then

(g) $$\left| p_N(X_n) - \frac{N_s}{nN}\, p_{(N_s/n)}(Y_{n,s}) \right.$$
$$\left. - \frac{N - \nu_{s+1}/n}{N}\, p_{(N-\nu_{s+1}/n)}(Y_{n,s+1}) \right| \leqq (\nu_{s+1} - N_s)/nN < \epsilon_s/3.$$

Combining (a), (b), and (g) we get

(h) $$\left| p_N(X_n) - \frac{nN - \nu_s}{nN}\, p^{k\alpha} \right| < \left(1 + \frac{nN - \nu_s}{nN}\right)\epsilon_s/3 < 2\epsilon_s/3.$$

Combining (h) with the inequality

(i) $$\left| p^{k\alpha} - \frac{nN - \nu_s}{Nn}\, p^{k\alpha} \right| < \epsilon_s/3$$

we get

(j) $\quad | p_N(X_n) - p^{ka} | < \epsilon_s$ if $(\nu_{s+1} + M_s)/n \leqq N \leqq (\nu_{s+2})/n$.

From (f) and (j) it follows that $P(X_n) = p^{ka}$ and since this equation holds for all possible numbers of the form, X_n, it follows that $x(a_1) \cdot x(a_2) \cdot x(a_3) \cdots x(a_a)$ belongs to $A(p^a)$. Hence the numbers, $x(a)$ (where $0<a<1$) constitute an admissibly independent set having the power of the continuum.

THEOREM 3. *There exists an admissibly independent set, I, such that every set, $I \cdot A(p)$, where $0 < p < 1$, has the power of the continuum.*

The set, I, will consist of the numbers $x(a, p)$, where $0 < a < 1$ and $0 < p < 1$, and where $x(a, p)$ belongs to the set $A(p)$. We have to show that if $a_1, a_2, a_3, \cdots, a_\gamma, p_1, p_2, \cdots, p_\gamma$ is any set of numbers such that $0 < a_1, a_2, \cdots, a_\gamma, p_1, \cdots, p_\gamma < 1$ and such that the equations, $a_i = a_j$ and $p_i = p_j$ are not both satisfied for any two distinct indices i and j, then $x(a_1, p_1) \cdot x(a_2, p_2) \cdots x(a_\gamma, p_\gamma)$ belongs to $A(p_1 \cdot p_2 \cdots p_\gamma)$.

We shall make use of an auxiliary set of numbers, $y^s{}_{\alpha,\beta}$, where $y^s{}_{\alpha,\beta}$ belongs to $A(\beta/2^s)$ and where $\alpha = 1, 2, 3, \cdots, s$ and $\beta = 1, 2, 3, \cdots, (2^s - 1)$. We shall define these numbers by means of the following equations:

$$y^s{}_{\alpha,\beta} = \sum_{i=1}^{\beta} \{ \prod_{j=1}^{r_i} (s(\alpha - 1) + s^2(\beta - 1) + r_{ij}/2^s s^2) y \cdot$$

$$\prod_{j=r_i+1}^{s} [1 - (s(\alpha - 1) + s^2(\beta - 1) + r_{ij}/2^s s^2) y] \},$$

where y belongs to $A(\frac{1}{2})$, $0 < r_{ij} \leqq s$, $r_{ij} \neq r_{ij'}$ if $j \neq j'$, and where the terms of the summation represent mutually exclusive event histories. It will be observed that $s(\alpha - 1) + s^2(\beta - 1) + r_{ij} < 2^s s^2$ and hence $y^s{}_{\alpha,\beta}$ belongs to $A(\beta/2^s)$.* Next let us consider the product, $y^s{}_{\alpha_1,\beta_1} \cdot y^s{}_{\alpha_2,\beta_2} \cdot y^s{}_{\alpha_3,\beta_3} \cdots y^s{}_{\alpha_m,\beta_m}$ where $m \leqq s$ and where the equations $\alpha_i = \alpha_j$ and $\beta_i = \beta_j$ are not both satisfied for any two distinct indices, i and j. This product can be expanded by means of the laws of ordinary algebra and we obtain an expression of the form

$$\sum_{i=1}^{\beta_1 \cdot \beta_2 \cdots \beta_m} \{ \prod_{j=1}^{R_i} (R_{i,j}/2^s s^2) y \cdot \prod_{j=r_i+1}^{sm} [1 - (R_{i,j}/2^s s^2) y] \},$$

where $0 < R_{i,j} < 2^s s^2$, $R_{i,j} \neq R_{i,j''}$ if $j \neq j'$ and where the terms of the summation represent mutually exclusive event histories. Hence $y^s{}_{\alpha_1,\beta_1} \cdot y^s{}_{\alpha_2,\beta_2} \cdots y^s{}_{\alpha_m,\beta_m}$ belongs to $A(\beta_1 \cdot \beta_2 \cdots \beta_m/2^{sm})$. Let $Y_{n,s} = (r_1/n)\{y^s{}_{\alpha_1,\beta_1} \cdots y^s{}_{\alpha_m,\beta_m}\} \cdot (r_2/n)\{y^s{}_{\alpha_1,\beta_1} \cdots\} \cdots (r_k/n)\{y^s{}_{\alpha_1,\beta_1} \cdots\}$ where $r_i \neq r_j$ if $i \neq j$ and where $0 < r_i \leqq n \leqq s$. Then $p(Y_{n,s}) = (\beta_1 \cdot \beta_2 \cdots \beta_m/2^{sm})^k$.

* See theorem 16 of the memoir cited.

Let $b_1, b_2, b_3, \cdots, b_m$ be a set of numbers such that $0 < b_i < 1$, and let β_i be the integer such that $(\beta_i + 1)/2^s > b_i \geqq \beta_i/2^s$. (In particular we could set $m = \gamma$, and $b_i = p_i$.) Then

$$\begin{aligned}(b_1 \cdot b_2 \cdots b_m)^k - (\beta_1 \cdot \beta_2 \cdots \beta_m/2^{sm})^k \\ < ((\beta_1 + 1)(\beta_2 + 1) \cdots (\beta_m + 1)/2^{sm})^k \\ - (\beta_1 \cdot \beta_2 \cdots \beta_m/2^{sm})^k.\end{aligned}$$

If $0 < \beta_i \leqq 2^s - 1$, $0 < k \leqq s$, and $0 < m \leqq s$, then the maximum of $((\beta_1 + 1) \cdots (\beta_m + 1/2^{sm})^k - (\beta_1 \cdot \beta_2 \cdots \beta_m/2^{sm})^k$ occurs when $\beta_1 = \beta_2 = \cdots = \beta_m = 2^s - 1$ and $k = m = s$ (s being fixed). Therefore

$$(b_1 \cdot b_2 \cdots b_m)^k - \beta_1 \cdot \beta_2 \cdots \beta_m)^k < 1 - (1 - 2^{-s})^{s^2}.$$

Let $\qquad \epsilon_s = 6[1 - (1 - 2^{-s})^{s^2}] \qquad$ then $\lim\limits_{s\to\infty} \epsilon_s = 0.$

We can define two sets of integers, $M_1, M_2, \cdots, M_s, \cdots$, and $N_1, N_2, \cdots, N_s, \cdots$, such that

(a) $\quad | p_N(Y_{n,s+1}) - (\beta_1 \cdot \beta_2 \cdots \beta_m/2^{(s+1)m})^k | < \epsilon_s/6 \qquad$ if $N \geqq M_s/n$

(b) $\quad | p_N(Y_{n,s}) - (\beta_1 \cdot \beta_2 \cdots \beta_m/2^{sm})^k | + \nu_s + M_s/N_s < \epsilon_s/6 \qquad$ if $N \geqq N_s/n$

where $\nu_s = N_1 + N_2 + \cdots + N_{s-1}$. The numbers M_s and N_s are such that conditions (a) and (b) hold simultaneously for all numbers, $Y_{n,s}$, for which $0 < \beta_i \leqq 2^s - 1$, $0 < \alpha_i$, $m \leqq s$ and $0 < k \leqq n \leqq s$. At the same time the numbers, N_s, are chosen so that ν_s/n is an integer if $n \leqq s$.

We can now define the numbers, $x(a, p)$, as follows: *The digits* $\nu_s + 1$ *to* $\nu_s + 1$ *of* $x(a, p)$ *are the same as digits* 1 *to* N_s of $y^s{}_{\alpha^s,\beta^s}$, *where* $\alpha^s = q_s(a)$ *and* $\beta^s = q_{2^s}(p)$.

Let us consider the number, $Y_{n,s}$, where $m = \gamma$, $\alpha_i{}^s = q_s(a_i)$, and $\beta_i{}^s = q_{2^s}(p_i)$, and let s_0 be such that the equations, $\alpha_i{}^s = \alpha_j{}^s$ and $\beta_i{}^s = \beta_j{}^s$ are not both satisfied if $s \geqq s_0$, and $i \neq j$. Combining (a) and (b) with the inequality

$$| (\beta_1{}^s \cdot \beta_2{}^s \cdots \beta_m{}^s/2^{\gamma s})^k - (p_1 p_2 p_3 \cdots p_\gamma)^k | < \epsilon_s/6$$

we get

(c) $\quad | p_N(Y_{n,s+1}) - (p_1 p_2 p_3 \cdots p_\gamma)^k | < \epsilon_s/3 \qquad$ if $N \geqq M_s/n$

(d) $\quad | p_N(Y_{n,s}) - (p_1 p_2 \cdots p_\gamma)^k | + (\nu_s + M_s)/N_s < \epsilon_s/3 \qquad$ if $N \geqq N_s/n$

whenever $s \geqq s_0$.

Let X_n be defined by the equation

$$\begin{aligned}X_n = (r_1/n)\{x(a_1, p_1) \cdot x(a_2, p_2) \cdots x(a_\gamma, p_\gamma)\} \cdot \{r_2/n) x(a_1, p_1) \cdots\} \\ \cdots (r_k/n)\{x(a_1, p_1) \cdots\}.\end{aligned}$$

Then it is easily seen that $| p_N(X_n) - (p_1 p_2 p_3 \cdots p_\gamma)^k | < \epsilon_s$, if $s \geqq s_0$ and if $\nu_{s+1}/n \leqq N \leqq \nu_{s+2}/n$. Therefore $p(X_n) = (p_1 p_2 \cdots p_\gamma)^k$. It follows that $x(a_1, p_1) \cdot x(a_2, p_2) \cdots x(a_\gamma, p_\gamma)$ belongs to $A(p_1 p_2 p_3 \cdots p_\gamma)$ and hence the numbers, $x(a, p)$, constitute an admissibly independent set.

We have now established the existence of non-denumerably infinite sets of admissibly independent numbers. One might ask whether the added condition of admissibility enables us to lighten the independence restriction. That is, can we say that n numbers are *admissibly* independent provided each is *admissibly* independent of each of the other $n - 1$? We have to answer this question in the negative as the following example shows. Let x belong to $A(\frac{1}{2})$ and let $x_1 = (1/2)x$, $x_2 = (2/2)x$ and $x_3 = (1/2)x \cdot (2/2)x + [1 - (1/2)x] \cdot [1 - (2/2)x]$. Then $x_1 \cdot x_2 = x_2 \cdot x_3 = x_3 \cdot x_1 = (1/2)x \cdot (2/2)x$ and $x_1 \cdot x_2 \cdot x_3 = (1/2)x \cdot (2/2)x$. Hence each of the numbers, x_1, x_2, x_3, is admissibly independent of each of the other two, but the set, x_1, x_2, x_3, is not admissibly independent.

THE RICE INSTITUTE,
HOUSTON, TEXAS.

The Primitive Groups of Class Fourteen.

By W. A. Manning.

1. No primitive group of class 14 contains a substitution of degree 14 and order 7.* Then the primitive groups of class 14 can be studied to the best advantage under two heads: (a) those in which the positive subgroup is of class > 15, to be determined in §§ 3-36 of this paper, (b) all others, deferred to §§ 37-48. It will be shown that there are but four groups of the first sort, and one of the second:

(1) *The simply transitive primitive group according to which the symmetric group of degree 9 permutes its 36 transpositions.*

(2) *The simply transitive primitive group of degree 49 and order* $2(7!)^2$ *isomorphic to* $\{ab,\ bcdefg,\ \alpha\beta,\ \beta\gamma\delta\epsilon\zeta\eta,\ a\alpha\cdot b\beta\cdot c\gamma\cdot d\delta\cdot e\epsilon\cdot f\zeta\cdot g\eta\}$.

(3) *The triply transitive group of degree 22 and order* $22\cdot 21\cdot 20\cdot 96$ *which occurs in Mathieu's quintuply transitive group of degree 24 as a transitive constituent of the largest subgroup in which the subgroup that leaves two letters fixed is invariant.*

(4) *The doubly transitive subgroup of (3) of degree 21 and order* $21\cdot 20\cdot 96$.

(5) *The doubly transitive group of degree 21 and order* $21\cdot 20\cdot 288$ *which occurs in the quintuply transitive group of degree 24 as a transitive constituent of the largest subgroup in which the subgroup that leaves three letters fixed is invariant.*

The positive subgroup of (5) is a doubly transitive group of class 15, which was omitted from the author's published list of the primitive groups of class 15.† The error in that paper is corrected in §§ 39-48. A direct proof of the existence of the quintuply transitive group of degree 24 is given in § 46.

2. In this investigation the following theorem is of constant use. It is, however, only a special case of a more general theory.‡

Let certain substitutions of prime order generate an invariant subgroup of a primitive group G. If H is an intransitive subgroup generated by some

* *Transactions of the American Mathematical Society*, Vol. 4 (1903), p. 351.

† *American Journal of Mathematics*, Vol. 39 (1917), p. 281.

‡ *Transactions of the American Mathematical Society*, Vol. 12 (1911), p. 378.

of these substitutions, there is in G a substitution similar to one of the given generators which replaces a letter of an arbitrarily chosen transitive constituent of H by a letter of another constituent of H.

Occasional use will be made of one of Jordan's theorems, which may be stated as follows: *

Let G be a transitive group generated by certain substitutions of prime order p, some of which generate an intransitive subgroup H. Let there be in G a substitution similar to one of the given generators of G which replaces a letter of an arbitrarily chosen transitive constituent A of H by a letter of some other constituent of H. Of such substitutions let t, $t_1, \cdots$ be those that displace a minimum number of letters new to H. Then if every power of some one of them, say t, replaces a letter of A by a letter of A, there is at most one letter new to H in any cycle of t.

The last condition of this theorem is certainly satisfied if the generating substitutions are of order 2 and if the degree of A exceeds the number of cycles in t.

The primitive groups of degree less than 21 are known and among them there are no primitive groups of class 14.†

3. If the positive subgroup of the primitive group G is of class > 15, a substitution s_2 of order 2 and degree 14 will with

$$s_1 = a_1a_2 \cdot b_1b_2 \cdot c_1c_2 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2$$

generate an Abelian group or one of the following non-Abelian diedral groups:

$$\begin{aligned}
D_0 &= \{s_1,\ s_2 = a_1b_1 \cdot c_1\alpha_1 \cdot c_2\alpha_2 \cdot d_1\beta_1 \cdot d_2\beta_2 \cdot e_1\gamma_1 \cdot e_2\gamma_2\},\\
D'_0 &= \{s_1,\ s_2 = a_1b_1 \cdot c_1d_1 \cdot e_1f_1 \cdot g_1\alpha_1 \cdot g_2\alpha_2 \cdot \beta_1\beta_2 \cdot \gamma_1\gamma_2\},\\
D_1 &= \{s_1,\ s_2 = a_1a_2 \cdot b_1c_1 \cdot d_1e_1 \cdot f_1\alpha_1 \cdot f_2\alpha_2 \cdot g_1\beta_1 \cdot g_2\beta_2\},\\
D_2 &= \{s_1,\ s_2 = a_1b_1 \cdot c_1d_1 \cdot e_1\alpha_1 \cdot e_2\alpha_2 \cdot f_1\beta_1 \cdot f_2\beta_2 \cdot \gamma_1\gamma_2\},\\
D_3 &= \{s_1,\ s_2 = a_1a_2 \cdot b_1b_3 \cdot c_1c_3 \cdot d_1d_3 \cdot e_1e_3 \cdot f_1f_3 \cdot g_1g_3\},\\
D_4 &= \{s_1,\ s_2 = a_1a_3 \cdot b_1b_3 \cdot c_1c_3 \cdot d_1d_3 \cdot e_1e_3 \cdot f_1f_3 \cdot g_1g_3\},\\
D_5 &= \{s_1,\ s_2 = a_1b_1 \cdot a_2c_1 \cdot b_2c_2 \cdot d_1d_3 \cdot e_1e_3 \cdot f_1f_3 \cdot g_1g_3\},\\
D_6 &= \{s_1,\ s_2 = a_1b_1 \cdot a_2c_1 \cdot b_2d_1 \cdot c_2d_2 \cdot e_1f_1 \cdot g_1\alpha_1 \cdot g_2\alpha_2\},\\
D_7 &= \{s_1,\ s_2 = a_1b_1 \cdot a_2c_1 \cdot b_2d_1 \cdot e_1f_1 \cdot e_2g_1 \cdot f_2\alpha_1 \cdot g_2\alpha_2\},
\end{aligned}$$

The accuracy of this list can be checked by the construction of the positive substitutions of degree > 15 and < 22 ‡ which can occur in G. D'_0 is the same group as D_0, with s_1 and s_2 transposed.

* C. Jordan, *Crelle's Journal*, Vol. 79 (1874), p. 249.

† See references, *American Journal of Mathematics*, Vol. 35 (1913), p. 229.

‡ *Transactions of the American Mathematical Society*, Vol. 18 (1917), p. 473.

Since no invariant subgroup of G can be Abelian unless it is regular, some two substitutions of a complete set of conjugate substitutions of degree 14 must generate one of these 8 diedral groups. Our immediate object is to show that G can not contain D_1, D_3, D_6, or D_7. They will be studied in the order D_7, D_3, D_6, D_1, D_5, D_0, D_2, D_4. A complete list of the possible Abelian groups generated by two substitutions of degree 14 might be given at this point, but there is nothing to be gained thereby. All such groups will be found in §§ 29-33.

4. The last group of this list, D_7, is of order 16 and has only two transitive constituents. It will be shown that it is not a subgroup of any primitive group of the sort we are studying. The following substitutions of D_7 will be needed:

$$\begin{aligned} s_1 &= a_1a_2 \cdot b_1b_2 \cdot c_1c_2 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2, \\ s_2 &= a_1b_1 \cdot a_2c_1 \cdot b_2d_1 \cdot e_1f_1 \cdot e_2g_1 \cdot f_2\alpha_1 \cdot g_2\alpha_2, \\ t_1 &= a_2d_1 \cdot b_2c_1 \cdot c_2d_2 \cdot e_1g_1 \cdot e_2f_1 \cdot f_2\alpha_2 \cdot g_2\alpha_1, \\ t_2 &= a_1b_2 \cdot a_2b_1 \cdot c_1d_2 \cdot c_2d_1 \cdot f_1g_2 \cdot f_2g_1 \cdot \alpha_1\alpha_2. \end{aligned}$$

There is a substitution s_3 of degree 14 that replaces a_1 by a letter of the second constituent and has at most one letter new to D_7 in any cycle. Under transformation by t_1 and $e_1e_2 \cdot f_1g_1 \cdot f_2g_2 \cdot \alpha_1\alpha_2$, the four transpositions (a_1e_1), (a_1e_2), (a_1f_1), and (a_1g_1) are conjugate. So also are (a_1f_2), (a_1g_2), $(a_1\alpha_1)$, and $(a_1\alpha_2)$. For s_3 we have to try only $(a_1e_1)\cdots$ and $(a_1f_2)\cdots$.

Since we are not interested in primitive groups of degree < 21,* it is well to know if a transitive group of degree 16 or 18 of which D_7 is a subgroup can be multiply imprimitive.† It is easy to verify the following statements:

The only possible system of imprimitivity of two letters including a_1 of any imprimitive group of degree 16 or 18 of which D_7 is a subgroup is a_1b_1.

The only possible system of imprimitivity of four letters including a_1 of any imprimitive group of degree 16 of which D_7 is a subgroup is $a_1b_1c_2d_2$.

No systems of three or nine letters are possible in an imprimitive group of degree 18 of which D_7 is a subgroup.

Then D_7 is a subgroup of a primitive group of degree < 21 unless s_3 displaces at least four letters new to D_7.

Neither $\{s_1, s_3\}$, $\{s_2, s_3\}$, $\{t_1, s_3\}$, or $\{t_2, s_3\}$ is Abelian or of the form D_6 or D_7. Nor can the first two be D_3 or D_4.

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* See references, *American Journal of Mathematics*, Vol. 35 (1913), p. 229.

† C. Jordan, *Liouville's Journal*, Ser. 2, Vol. 16 (1871), p. 383; Marggraff, Dissertation, *Ueber primitive Gruppen mit transitiven Untergruppen geringeren Grades*, Giessen, 1889; Manning, *Primitive Groups*, 1921, p. 92.

5. Consider first the substitution

$$s_3 = (a_1e_1)(\beta_1 —)(\beta_2 —)(\beta_3 —)(\beta_4 —)\cdots.$$

Let both $\{s_1, s_3\}$ and $\{s_2, s_3\}$ be octic. Now $s_3 = (a_1e_1)(a_2)(b_1)\cdots$ so that $\{t_2, s_3\}$ is octic and $s_3 = (a_1e_1)(b_2\beta_1)(b_1)\cdots$, which is impossible when $\{s_1, s_3\}$ is octic.

Let $\{s_1, s_3\}$ be octic and $\{s_2, s_3\}$ such a group as D_5. Now $s_3 = (a_1e_1)(b_1x)(f_1y)(\beta_1 —)\cdots(a_2)(e_2)$, where (xy) is (b_2d_1), $(f_2\alpha_1)$, or $(g_2\alpha_2)$. If $\{s_1, s_3\}$ is D_1, $s_3 = (a_1e_1)(c_1\beta_1)(c_2\beta_2)(a_2)(e_2)\cdots$, which is impossible with s_2. Therefore $\{s_1, s_3\}$ is D_0 or D_2, and $s_3 = (a_1e_1)(b_1d_1)(f_1b_2)\cdots$ or $(a_1e_1)(f_1\alpha_1)(b_1f_2)\cdots$, both of which are impossible; or else $xy = g_2\alpha_2$, in which case e_2 and g_1 are fixed, impossible with s_2.

If $\{s_1, s_3\}$ is D_5 and $\{s_2, s_3\}$ is octic, transformation by $a_1e_1 \cdot a_2f_1 \cdot b_1e_2 \cdot b_2g_1 \cdot c_1f_2 \cdot c_2\alpha_1 \cdot d_1g_2 \cdot d_2\alpha_2$ shifts this case back to the preceding.

Let $\{s_1, s_3\}$ and $\{s_2, s_3\}$ be D_5. It is not possible to have $s_3 = (a_1e_1)(a_2x)(e_2y)\cdots = (a_1e_1)(b_1u)(f_1v)\cdots$.

6. Consider second $s_3 = (a_1f_2)\cdots$.

Let $\{s_1, s_3\}$ and $\{s_2, s_3\}$ be octic. Since now s_3 fixes a_2 and b_1, $\{t_2, s_3\}$ is D_0 or D_2. Therefore $s_3 = (a_1f_2)(a_2)(b_1)(b_2)(f_1)(g_1)(\alpha_1)\cdots$, and $s_3 \neq (\alpha_2\beta_1)(\alpha_1)\cdots$. Thus s_3 has no cycle new to s_1, and in consequence $\{s_1, s_3\}$ is D_0. Then s_3 fixes g_2, so that $\{t_2, s_3\}$ is impossible.

Let $\{s_1, s_3\}$ be octic and $\{s_2, s_3\}$ of order 6. Now $s_3 = (a_1f_2)(b_1x)(\alpha_1y)(a_2)(f_1)\cdots$, where the cycle (xy) of s_2 is (b_2d_1), (e_2g_1), or $(g_2\alpha_2)$. Moreover, $x \neq g_1$, e_1, or e_2, and $y \neq d_1$, d_2, c_1, or c_2, for otherwise $(b_1x)(\alpha_1y)\cdots$ can be transformed into $(a_1e_1)\cdots$. We cannot have $s_3 = (a_1f_2)(b_1d_1)(\alpha_1b_2)\cdots$, and neither $(a_1f_2)(b_1g_2)(\alpha_1\alpha_2)\cdots$ nor $(a_1f_2)(b_1\alpha_2)(\alpha_1g_2)\cdots$ is possible.

Let $\{s_1, s_3\}$ be D_5 and $\{s_2, s_3\}$ octic. As before, $s_3 = (a_1f_2)(a_2x)(f_1y)(b_1)(\alpha_1)\cdots$, where (xy) is (c_1c_2), (d_1d_2), (e_1e_2), or (g_1g_2), and $x \neq e_1$, e_2, or g_2, $y \neq b_1$, c_1, or d_1. The three forms of s_3, $(a_1f_2)(a_2c_1)(f_1c_2)\cdots$, $(a_1f_2)(a_2d_1)(f_1d_2)\cdots$ and $(a_1f_2)(a_2g_1)(f_1g_2)\cdots$ are all impossible.

Finally, $\{s_1, s_3\}$ and $\{s_2, s_3\}$ cannot both be such groups as D_5.

Therefore D_7 can be struck from our list.

7. Consider a group G in which D_3 is a subgroup. If G exists there is in it a substitution $s_3 = (a_1b_1)\cdots$. Now s_3 is not $(a_1b_1)(a_2b_2)\cdots$. Nor is $s_3 = (a_1b_1)(a_2c_1)(b_2c_2)\cdots$ or $(a_1b_1)(a_2c_2)\cdots$. Then $\{s_1, s_3\}$ and $\{s_2, s_3\}$ must be octic groups. If s_3 fixes a_2, it fixes also b_2 and b_3, so that $\{s_2s_1s_2, s_3\}$ is an impossibility. If s_3 displaces a_2, $\{s_1, s_3\}$ and $\{s_2, s_3\}$ are both D_6, $s_3 =$

$(a_1b_1)(a_2c_1)(b_2d)(c_2d)(b_3 —) \cdots$, and $\{s_1s_2s_1, s_3\}$ is D_0, D'_0, D_1, or D_2. Hence $s_3 = (a_1b_1)(a_2c_1)(b_2d_1)(c_2d_2)(b_3 —)(c_3) \cdots$. But if $\{s_2, s_3\}$ is D_6, s_3 should displace c_3. *This consideration removes D_3 from our list.*

8. We can also rid our list of D_6. Its substitutions of degree 14 are

$$\begin{aligned}
s_1 &= a_1a_2 \cdot b_1b_2 \cdot c_1c_2 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2,\\
s_2 &= a_1b_1 \cdot a_2c_1 \cdot b_2d_1 \cdot c_2d_2 \cdot e_1f_1 \cdot g_1\alpha_1 \cdot g_2\alpha_2,\\
t_1 &= a_1d_1 \cdot a_2d_2 \cdot b_1c_1 \cdot b_2c_2 \cdot e_1f_2 \cdot e_2f_1 \cdot \alpha_1\alpha_2,\\
t_2 &= a_1c_2 \cdot a_2b_2 \cdot b_1d_2 \cdot c_1d_1 \cdot e_2f_2 \cdot g_1\alpha_2 \cdot g_2\alpha_1,
\end{aligned}$$

All the substitutions of the group $\{\sigma = e_1f_1 \cdot e_2f_2,\ \sigma_1 = g_1g_2 \cdot \alpha_1\alpha_2,\ \sigma_2 = b_1c_2 \cdot b_2c_1 \cdot e_1e_2 \cdot f_1f_2 \cdot \alpha_1\alpha_2,\ \sigma_3 = a_2b_1 \cdot b_2c_1 \cdot c_2d_1 \cdot e_1g_1 \cdot e_2\alpha_1 \cdot f_1g_2 \cdot f_2\alpha_2\}$ transform D_6 into itself. There is then but one substitution s_3 to be investigated: $(a_1e_1) \cdots$. Upon this substitution may be imposed the condition that it introduces a minimum number of new letters to D_6, and therefore at most one new letter to any cycle.

If $\{D_6, s_3\}$ $(= H_3)$ is an imprimitive group of degree 16 or 18, the only system, including α_1, of two letters is $\alpha_1\alpha_2g_1g_2$, and there can be no system of three letters. Hence if H_3 is a transitive group of degree < 20, D_6 is a subgroup of a primitive group of degree < 21. That is, if H_3 is of degree < 20, it is intransitive.

9. The group $\{s_1, s_3\}$ can be Abelian, or of one of the types D_0, D'_0, D_2, D_5, or D_6.

Let $\{s_1, s_3\}$ be Abelian. If s_3 and t_1 are also commutative, $s_3 = (a_1e_1)(a_2e_2)(d_1f_2)(d_2f_1) \cdots$. By s_2, $s_3 = (a_1e_1)(f_1d_2)(b_1c_2)(a_2e_2)(c_1) \cdots$. Then $\{t_1, s_3\}$ is octic. If $\{t_1, s_3\}$ is D_6, there are just two letters in s_3 new to t_1, and they replace the letters of one cycle of t_1. These new letters are not g_1 or g_2. Therefore $s_3 = (a_1e_1)(a_2e_2)(\alpha_1\beta_1)(\alpha_2\beta_2) \cdots$. The only other letter new to s_2 in s_3 is now f_2, so that $\{s_2, s_3\}$ can only be D_1 or D_5, and neither case is possible. If $\{t_1, s_3\}$ is D'_0, D_1, or D_2, s_3 has four or six letters new to t_1. Now $\{s_2, s_3\}$ is octic because s_3 fixes f_1 and therefore $s_3 = (a_1e_1)(a_2e_2)(c_1 —)(b_1)(b_2)(d_1)(d_2)(f_1)(f_2) \cdots$, and this is impossible.

Let $\{s_1, s_3\}$ be D_0. The substitution $s_3 = (a_1e_1)(a_2)(e_2) \cdots$ has six letters new to s_1 in six cycles, of which at least four are β_1, β_2, β_3, β_4. If s_3 displaces α_1 and α_2, $\{t_1, s_3\}$ is D_0, D_1, D_2, or D_5. If it is D_5, $\{s_2, s_3\}$ is impossible; and if it is D_0, D_1, or D_2, s_3 fixes a_2, e_2, d_1, d_2, f_1, f_2, and therefore $s_3 = (a_1e_1)(b_1\beta_1)(b_2\beta_2)(c_1\beta_3)(c_2\beta_4)(\alpha_1g)(\alpha_2g) \cdots$, absurd with s_2. If s_3 displaces α_1 and fixes α_2, $\{t_1, s_3\}$ is D_2 and $s_3 = (a_1e_1)(g_1\beta_1)(g_2\beta_2)(\alpha_1x) \cdots$, where x is new to s_2 and to t_2 and is a letter of s_1, an impossibility. If s_3 fixes

α_1 and α_2, it displaces six letters β in six cycles, so that $\{t_1, s_3\}$ and $\{s_2, s_3\}$ are also D_0. This is impossible.

Let $\{s_1, s_3\}$ be D'_0. Now s_3 has six letters new to s_1 and four of these new letters are in two cycles. Therefore $s_3 = (a_1e_1)(\alpha_1\beta_1)(\alpha_2\beta_2)(\beta_3x)(\beta_4y)\cdots$, where (xy) is a cycle of s_1. We note that $\{t_1, s_3\}$ is octic and is not D_6. Therefore s_3 fixes a_2, e_2, d_1, f_2, and (xy) is (b_1b_2), (c_1c_2), or (g_1g_2). Because of s_2, (xy) is not (b_1b_2). If (xy) is (c_1c_2), t_1 requires that b_1 and b_2 be displaced, and this s_1 does not permit. If (xy) is (g_1g_2), $\{s_2, s_3\}$ is octic and therefore s_3 fixes f_1 and f_2, whereas s_1 requires that f_1 or f_2 be displaced.

Let $\{s_1, s_3\}$ be D_1. Here s_3 has a cycle in common with s_1 and has four letters new to s_1. If s_3 displaces α_1 it is of the form $(a_1e)(\alpha_1g)(\alpha_2g)\cdots$. For if H_3 is of degree < 20, it is intransitive. This form of s_3 is inconsistent with s_2 or t_2. Let s_3 replace g_1 by g_2. Thus $s_3 = (a_1e_1)(g_1g_2)(\beta_1 —)(\beta_2 —)(\beta_3 —)(\beta_4 —)\cdots$. With s_2, t_1 and t_2, s_3 generates octic groups and therefore fixes f_1 and f_2, but should displace a letter from every cycle of s_1. Let $s_3 = (a_1e_1)(g_1\beta_1)(g_2\beta_2)\cdots$. This is commutative with t_1 and replaces d_1 by f_2. Hence if s_3 has one cycle in common with s_1, it has two.

Let $\{s_1, s_3\}$ be D_2. Now s_3 has six letters new to s_1, in five cycles. If $s_3 = (a_1e_1)(\alpha_1\beta_1)(\alpha_2 —)(\beta_2 —)(\beta_3 —)(\beta_4 —)\cdots$, the groups $\{t_1, s_3\}$, $\{s_2, s_3\}$ and $\{t_2, s_3\}$ are octic, so that $s_3 = (a_1e_1)(\alpha_1\beta_1)(g\alpha_2)(g\beta_2)\cdots$, impossible with s_2 and t_2. If $s_3 = (a_1e_1)(\alpha_1\beta_1)(\beta_2 —)(\beta_3 —)(\beta_4 —)(\beta_5 —)\cdots$, $\{t_1, s_3\}$ is D_5; but $(a_1e_1)(\alpha_1\beta_1)(d_1 —)(f_2 —)\cdots$ is absurd if the blanks are to be filled from t_1. It would require eight cycles in s_3. Then $s_3 = (a_1e_1)(\alpha_1\alpha_2)(\beta_1 —)(\beta_2 —)(\beta_3 —)(\beta_4 —)\cdots$ and $s_3 \neq (g_1g_2)\cdots$. Therefore s_3 is not commutative with s_2 or t_2. Both $\{s_2, s_3\}$ and $\{t_1, s_3\}$ are octic. Hence s_3 fixes f_1, f_2, g_1 and g_2.

10. Let $\{s_1, s_3\}$ be D_5. Now $s_3 = (a_1e_1)(a_2x)(e_2y)(\beta_1 —)(\beta_2 —)\cdots$, where (xy) is a cycle of s_1. In s_3 there are four letters new to s_1. If one of them is α_1, H_3 is intransitive and therefore $s_3 = (\alpha_1g)(\alpha_2)(g)\cdots = (a_1e_1)(d_1x_1)(f_2y_1)\cdots$ [(x_1y_1) is a cycle of t_1], which is impossible. Hence $s_3 = (\alpha_1)(\alpha_2)(a_1e_1)(a_2x)(e_2y)(\beta_1 —)(\beta_2 —)(\beta_3 —)(\beta_4 —)$. Since $\{t_1, s_3\}$ is D_0 or D_2, s_3 fixes d_1 and f_2, and then, because of s_1, must replace d_2 and f_1 by letters β. But $s_3 = (e_1a_1)(f_1\beta_1)\cdots$ is impossible with s_2.

Let $\{s_1, s_3\}$ be D_6. H_3 is intransitive. Then $s_3 = (a_1e_1)(g_1\beta_1)(g_2\beta_2)\cdots$ or $(g_1\alpha)(g_2x)\cdots$, where x is new to s_1. But $(g_1\alpha)(g_2\beta)(\alpha)\cdots$ is impossible, as is $(g_1\alpha)(g_2\alpha)\cdots$. Now $s_3 = (a_1e_1)(g_1\beta_1)(g_2\beta_2)(\alpha_1)(\alpha_2)(d_1f_2)\cdots$. Since $\{s_2, s_3\}$ and $\{t_2, s_3\}$ are of order 6, $s_3 = a_1e_1 \cdot a_2f_1 \cdot b_1c_1 \cdot d_1f_2 \cdot d_2e_2 \cdot g_1\beta_1 \cdot g_2\beta_2$.

The group $H_3 = \{D_6, s_3\}$ is unique. It is transformed into itself by $g_1g_2 \cdot \alpha_1\alpha_2 \cdot \beta_1\beta_2$ and by $t_3 = c_1e_2 \cdot c_2e_1 \cdot b_1f_2 \cdot b_2f_1 \cdot d_1d_2 \cdot \alpha_1\beta_2 \cdot \alpha_2\beta_1$, so that for s_4 two substitutions, $(a_1g_1) \cdots$ and $(a_1\alpha_1) \cdots$, are to be studied. Now $(a_1g_1) \cdots$ can only be the transform of s_3 above by the substitution $\sigma_3 = a_2b_1 \cdot b_2c_1 \cdot c_2d_1 \cdot e_1g_1 \cdot e_2\alpha_1 \cdot f_1g_2 \cdot f_2\alpha_2$ of the group whose substitutions transform D_6 into itself: $\sigma_3 s_3 \sigma_3 = a_1g_1 \cdot a_2b_2 \cdot b_1g_2 \cdot c_2\alpha_2 \cdot d_2\alpha_1 \cdot e_1x \cdot f_1y$, where x and y are letters new to D_6. This is impossible with s_3. The other substitution, $(a_1\alpha_1) \cdots$, is the transform of s_3 by $\sigma_2\sigma_3$: $\sigma_3\sigma_2s_3\sigma_2\sigma_3 = a_1\alpha_1 \cdot b_1\alpha_2 \cdot c_1d_1 \cdot c_2g_2 \cdot d_2g_1 \cdot e_1x \cdot f_1y$. This too is inconsistent with s_3.

This leaves in our list of diedral groups D_0, D'_0, D_1, D_2, D_4 and D_5.

11. It is also possible to reject the group D_1. The four negative substitutions of order 2 in D_1 are

$$\begin{aligned} s_1 &= a_1a_2 \cdot b_1b_2 \cdot c_1c_2 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2, \\ s_2 &= a_1a_2 \cdot b_1c_1 \cdot d_1e_1 \cdot f_1\alpha_1 \cdot f_2\alpha_2 \cdot g_1\beta_1 \cdot g_2\beta_2, \\ t_1 &= a_1a_2 \cdot b_2c_2 \cdot d_2e_2 \cdot f_2\alpha_1 \cdot f_1\alpha_2 \cdot g_2\beta_1 \cdot g_1\beta_2, \\ t_2 &= a_1a_2 \cdot b_1c_2 \cdot b_2c_1 \cdot d_1e_2 \cdot d_2e_1 \cdot \alpha_1\alpha_2 \cdot \beta_1\beta_2. \end{aligned}$$

All the substitutions of the group

$$\begin{aligned} \{&b_1b_2 \cdot c_1c_2 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2, \quad b_1c_1 \cdot d_1e_1 \cdot f_1\alpha_1 \cdot f_2\alpha_2 \cdot g_1\beta_1 \cdot g_2\beta_2, \\ &b_1d_1 \cdot b_2d_2 \cdot c_1e_1 \cdot c_2e_2, \quad b_1f_1 \cdot b_2\alpha_1 \cdot c_1f_2 \cdot c_2\alpha_2 \cdot d_1g_1 \cdot d_2\beta_1 \cdot e_1g_2 \cdot e_2\beta_2, \\ &b_1c_1 \cdot b_2c_2,\ d_1e_1 \cdot d_2e_2,\ f_1f_2 \cdot \alpha_1\alpha_2,\ g_1g_2 \cdot \beta_1\beta_2,\ f_1g_1 \cdot f_2g_2 \cdot \alpha_1\beta_1 \cdot \alpha_2\beta_2\} \end{aligned}$$

transform D_1 into itself. Then for s_3 only the substitution $(a_1b_1) \cdots$ need be studied.

If s_3 is commutative with s_1, $\{s_2, s_3\}$ is not one of our groups D_0, D'_0, D_1, D_2, D_4, or D_5. If $\{s_1, s_3\}$ is octic, s_3 fixes a_2, b_2, c_1 and c_2 and then $\{t_1, s_3\}$ is not a possible subgroup of G. Hence $\{s_1, s_3\}$ is D_5: $s_3 = (a_1b_1)(a_2 —)(b_2 —) \cdots$, and $(t_1s_3)^4 = (s_2s_3)^2 = 1$. Indeed, $\{t_1, s_3\}$ is D_1. Therefore

$$s_3 = a_1b_1 \cdot a_2c_1 \cdot b_2c_2 \cdot d_2\gamma_1 \cdot e_2\gamma_2 \cdot f_1\alpha_1 \cdot g_1\beta_1$$

or

$$s'_3 = a_1b_1 \cdot a_2c_1 \cdot b_2c_2 \cdot d_2\gamma_1 \cdot e_2\gamma_2 \cdot f_1\beta_1 \cdot g_1\alpha_1.$$

The only other substitutions of degree 14 which replace a_1 by b_1 are the transforms of s_3 and s'_3 by $f_1f_2 \cdot \alpha_1\alpha_2$, $g_1g_2 \cdot \beta_1\beta_2$ and $f_1f_2 \cdot g_1g_2 \cdot \alpha_1\alpha_2 \cdot \beta_1\beta_2$. The sets of transitivity of H_3 are $a_1a_2b_1b_2c_1c_2$, $d_1d_2e_1e_2\gamma_1\gamma_2$, $f_1f_2\alpha_1\alpha_2$, $g_1g_2\beta_1\beta_2$, and of H'_3 are $a_1a_2b_1b_2c_1c_2$, $d_1d_2e_1e_2\gamma_1\gamma_2$, $f_1f_2g_1g_2\alpha_1\alpha_2\beta_1\beta_2$. Both H_3 and H'_3 are invariant under the substitutions of the group $\{d_1e_1 \cdot d_2e_2 \cdot \gamma_1\gamma_2,\ b_1c_2 \cdot b_2c_1 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2,\ f_1g_1 \cdot f_2g_2 \cdot \alpha_1\beta_1 \cdot \alpha_2\beta_2,\ a_1a_2 \cdot b_1c_1 \cdot d_1e_1 \cdot f_1\alpha_1 \cdot f_2\alpha_2 \cdot g_1\beta_1 \cdot g_2\beta_2\}$. Hence for s_4 we have to discuss only the three substitutions $(a_1d_1) \cdots$, $(a_1\gamma_1) \cdots$ and $(a_1f_1) \cdots$.

It will be convenient to use w to represent s_3 and s'_3 indifferently.

12. Let $s_4=(a_1f_1)\cdots$. Take the eight substitutions $(a_1b_1)\cdots$ which are possible with $\{s_1, s_2\}$ and transform them by $b_1f_1\cdot b_2\alpha_1\cdot c_1f_2\cdot c_2\alpha_2\cdot d_1g_1\cdot d_2\beta_1\cdot e_1g_2\cdot e_2\beta_2$. Transform again by $d_1e_1\cdot d_2e_2\cdot\gamma_1\gamma_2$ and by $b_1c_2\cdot b_2c_1\cdot d_1d_2\cdot e_1e_2\cdot f_1f_2\cdot g_1g_2$. There remain for further consideration two substitutions having the first cycles $a_1f_1\cdot a_2f_2\cdot\alpha_1\alpha_2\cdot\beta_1x_1\cdot\beta_2x_2$ and the last two cycles $b_1b_2\cdot d_1d_2$ or $b_1d_2\cdot b_2d_1$. Of the letters x_1 and x_2 we know as yet merely that they are new to D_1. Now $s_4=(f_1a_1)(\alpha_1\alpha_2)\cdots$ is not possible with s_3. The group $\{s'_3, s_4\}$ is D_5, so that $s_4=a_1f_1\cdot a_2f_2\cdot b_1d_2\cdot b_2d_1\cdot\alpha_1\alpha_2\cdot\beta_1\gamma_1\cdot\beta_2\gamma_2$. In the primitive group G there is no other substitution $(a_1f_1)\cdots$ similar to s_1. This means that no G containing this H_4 is doubly transitive, for if so there would be four substitutions with (a_1f_1) in common, conjugate to s_1, s_2, t_1 and t_2, respectively. Since $\{H'_3, s_4\}$ is transitive of degree 20, and since there is no simply transitive primitive group of degree 20, this $s_4=(a_1f_1)\cdots$ can be dropped.

Let $s_4=(a_1d_1)\cdots$. Now there are six substitutions s_4 with $a_1d_1\cdot a_2e_1\cdot d_2e_2\cdot b_2x_1\cdot c_2x_2$ in common and with the last two cycles $f_1\alpha_1\cdot g_1\beta_1$, $f_1\alpha_1\cdot g_2\beta_2$, $f_2\alpha_2\cdot g_2\beta_2$, $f_1\beta_1\cdot g_1\alpha_1$, $f_1\beta_2\cdot g_2\alpha_1$, or $f_2\beta_2\cdot g_2\alpha_2$. This s_4 is commutative with $s_1s'_3s_1=a_1c_2\cdot a_2b_2\cdot b_1c_1\cdot d_1\gamma_1\cdot e_1\gamma_2\cdot f_2\beta_1\cdot g_2\alpha_1$. Thus we have uniquely $s_4=a_1d_1\cdot a_2e_1\cdot d_2e_2\cdot b_2\gamma_2\cdot c_2\gamma_1\cdot f_1\beta_2\cdot g_2\alpha_1$. The group $\{H_3, s_4\}$ is impossible. Of $H_4=\{H'_3, s_4\}$ it can at least be said that it has two sets of transitivity. Transformation by substitutions of D_1 and by $s'_3t_1s'_3=b_1c_1\cdot b_2c_2\cdot f_1g_2\cdot f_2g_1\cdot\alpha_1\beta_2\cdot\alpha_2\beta_1\cdot\gamma_1\gamma_2$ shows that there is but one substitution to try for s_5: $(a_1f_1)\cdots$, and it has already been rejected.

Let $s_4=(a_1\gamma_1)\cdots$, with the strong condition that in no substitution of G, similar to s_1, is a_1 (or a_2) replaced by d_1, d_2, e_1, or e_2. Let s_4 fix a_2. If $ws_4=s_4w$, $s_4=(a_1\gamma_1)(b_1d_2)\cdots$, and the transform of $t_1wt_1=(a_2b_1)\cdots$ by s_4 is $(a_2d_2)\cdots$. If $\{w, s_4\}$ is octic, $s_4=(a_1\gamma_1)(a_2)(b_1)(b_2\text{—})(c_1\text{—})(c_2)\cdots$, and is inconsistent with t_2. If $\{w, s_4\}$ is of order 6, $s_4=(a_1\gamma_1)(b_1\text{—})(d_2\text{—})(c_1\text{—})(a_2)(b_2)(d_1)(c_2)\cdots$, clearly impossible with t_1,

Now the only letter by which s_4 can replace a_2 is γ_2: $s_4=(a_1\gamma_1)(a_2\gamma_2)\cdots$. If $ws_4=s_4w$, $s_4=(a_1\gamma_1)(a_2\gamma_2)(b_1d_2)(b_2)\cdots$. Then $s_4s_1ws_1s_4=(b_2\gamma_2)\cdots$ and the transform of this by s_1ws_1 is $(a_2e_1)\cdots$. If $\{w, s_4\}$ is octic, $s_4=(a_1\gamma_1)(a_2\gamma_2)(b_1)\cdots$, and $ws_4ws_4w=(a_1d_2)\cdots$.

Then the only non-Abelian diedral groups left are D_0, D'_0, D_2, D_4 and D_5.

13. Consider D_5, in which

$$s_1=a_1a_2\cdot b_1b_2\cdot c_1c_2\cdot d_1d_2\cdot e_1e_2\cdot f_1f_2\cdot g_1g_2,$$
$$s_2=a_1b_1\cdot a_2c_1\cdot b_2c_2\cdot d_1d_3\cdot e_1e_3\cdot f_1f_3\cdot g_1g_3,$$
$$t_1=a_2b_2\cdot a_1c_2\cdot b_1c_1\cdot d_2d_3\cdot e_2e_3\cdot f_2f_3\cdot g_2g_2.$$

The group D_5 is invariant under $\{\sigma_1 = b_1c_2 \cdot b_2c_1 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2,\ \sigma_2 = a_2b_1 \cdot b_2c_1 \cdot d_2d_3 \cdot e_2e_3 \cdot f_2f_3 \cdot g_2g_3\}$ so that we have to study only the one type of substitution, $s_3 = (a_1d_1) \cdots$. If $\{s_1, s_3\}$ is an octic group, $\sigma_1 s_3 s_1 s_3 \sigma_1 = (a_1d_1)(a_2d_2) \cdots$, and if $\{s_2, s_3\}$ is octic, $\sigma_1\sigma_2 s_3 s_2 s_3 \sigma_2\sigma_1 = (a_1d_1)(a_2d_2) \cdots$. If $(s_2s_3)^2 = 1$, $\sigma_2 s_3 \sigma_2 = (a_1d_1)(a_2d_2) \cdots$. It is not possible to have all the groups $\{s_1, s_3\}$, $\{s_2, s_3\}$, $\{t_1, s_3\}$ of order 6, as is seen by considering one of the transitive constituents of degree 3. If $(s_1s_3)^3 = (s_2s_3)^3 = (t_1s_3)^4 = 1$, $s_3 = (a_1d_1)(c_2\alpha)(b_2)(a_2e_2)(e_3) \cdots$, and the transform of the cycles $(a_2b_2)(e_2e_3)$ of t_1 by $s_3s_1\sigma_2\sigma_1(d_1e_1)(d_2e_2)(d_3e_3)$ is $(a_1d_1)(a_2d_2)$. Then in all cases $s_3 = (a_1d_1)(a_2d_2) \cdots$. If $(s_2s_3)^4 = 1$, $s_3 = (a_1d_1)(a_2d_2)(b_1)(d_3) \cdots$, and $s_3\sigma_2\sigma_1$ transforms $s_2 = (a_1b_1)(d_1d_3)(a_2c_1) \cdots$ into $(d_2a_2)(a_1d_1)(d_3 \text{—}) \cdots$, which with s_2 generates such a group as D_5. Then

$$s_3 = a_1d_1 \cdot a_2d_2 \cdot b_1e_1 \cdot b_2e_2 \cdot d_3e_3 \cdot f_1f_2 \cdot g_3g_4$$

or

$$s'_3 = a_1d_1 \cdot a_2d_2 \cdot b_1e_1 \cdot b_2e_2 \cdot d_3e_3 \cdot f_3f_4 \cdot g_3g_4.$$

14. We take up the first of these two substitutions and study the group H_3. The substitutions of degree 14 in H_3 are

$$\begin{aligned}
s_1 &= a_1a_2 \cdot b_1b_2 \cdot c_1c_2 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2,\\
s_2 &= a_1b_1 \cdot a_2c_1 \cdot b_2c_2 \cdot d_1d_3 \cdot e_1e_3 \cdot f_1f_3 \cdot g_1g_3,\\
t_1 &= a_2b_2 \cdot a_1c_2 \cdot b_1c_1 \cdot d_2d_3 \cdot e_2e_3 \cdot f_2f_3 \cdot g_2g_3,\\
s_3 &= a_1d_1 \cdot a_2d_2 \cdot b_1e_1 \cdot b_2e_2 \cdot d_3e_3 \cdot f_1f_2 \cdot g_3g_4,\\
t_2 &= b_1d_3 \cdot c_1d_2 \cdot a_1e_3 \cdot c_2e_2 \cdot d_1e_1 \cdot f_2f_3 \cdot g_1g_4,\\
t_3 &= b_2d_3 \cdot c_2d_1 \cdot a_2e_3 \cdot c_1e_1 \cdot d_2e_2 \cdot f_1f_3 \cdot g_2g_4.
\end{aligned}$$

The sets of transitivity are $a_1a_2b_1b_2c_1c_2d_1d_2d_3e_1e_2e_3$, $f_1f_2f_3$, and $g_1g_2g_3g_4$. H_3 is invariant under $\sigma = a_2c_2 \cdot b_2c_1 \cdot d_2e_2 \cdot d_1e_3 \cdot d_3e_1 \cdot f_1f_3 \cdot g_1g_3$. There are then four types of substitutions that need be used for s_4: $(a_1f_1) \cdots$, $(a_1f_2) \cdots$, $(a_1g_1) \cdots$, and $(a_1g_2) \cdots$. If s_1s_4 is of order 4, $s_4s_1s_4$ is $(a_1f_2)(a_2f_1) \cdots$, $(a_1f_1)(a_2f_2) \cdots$, $(a_1g_2)(a_2g_1) \cdots$, or $(a_1g_1)(a_2g_2) \cdots$, and may be used for s_4. The products s_1s_4 and s_2s_4 are not both of order 2. Hence the substitution s_4 may be determined subject to the following conditions:

(a) $(s_1s_4)^2 = (s_2s_4)^3 = 1$,

(b) $(s_1s_4)^3 = (s_2s_4)^2 = 1$,

(c) $(s_1s_4)^3 = (s_2s_4)^3 = 1$,

(d) $(s_1s_4)^2 = (s_2s_4)^4 = 1$, $s_4 = (a_1f_2) \cdots$, or $(a_1g_2) \cdots$,

(e) $(s_1s_4)^3 = (s_2s_4)^4 = 1$, $s_4 = (a_1f_2) \cdots$, or $(a_1g_2) \cdots$.

To justify the limitation in (d) and (e), consider $s_4 = (a_1f_1)(b_1)(f_3) \cdots$. The transform $t = t_3s_4s_2s_4t_3 = (a_1f_1)(b_1f_3) \cdots$, and we may assume that ts_1 is not of order 4. For if it were of order 4, $ts_1t = (a_1f_2)(a_2f_1) \cdots$ could be

used for s_4, under (a), or under (d) as stated. If $s_4=(a_1g_1)(b_1)(g_3)\cdots$, we use $t=\sigma s_4 s_2 s_4 \sigma=(a_1g_1)(b_1g_3)\cdots$ and again choose another substitution $(a_1g_1)(a_2g_2)\cdots$ for s_4 if ts_1 is of order 4. It will be seen that not all the substitutions s_4 determined subject to the above conditions have to be used for s_4, but these five cases certainly include all the necessary forms of s_4. The following 19 substitutions result immediately from this classification:

$$
\begin{aligned}
S_{4\ 1} &= a_1f_1\cdot a_2f_2\cdot b_1g_1\cdot b_2g_2\cdot d_1d_2\cdot f_3g_3\cdot e_3\alpha\,,\\
S_{4\ 2} &= a_1f_2\cdot a_2f_1\cdot c_1e_1\cdot c_2e_2\cdot d_1d_2\cdot e_3f_3\cdot g_3\alpha\,,\\
S_{4\ 3} &= a_1g_1\cdot a_2g_2\cdot b_1f_1\cdot b_2f_2\cdot e_1e_2\cdot f_3g_3\cdot d_3\alpha\,,\\
S_{4\ 4} &= a_1g_2\cdot a_2g_1\cdot c_1d_1\cdot c_2d_2\cdot e_1e_2\cdot d_3g_3\cdot f_3\alpha\,,\\
S_{4\ 5} &= a_1f_1\cdot a_2d_1\cdot b_1f_3\cdot c_1d_3\cdot d_2f_2\cdot g_1g_3\cdot e_2\alpha\,,\\
S_{4\ 6} &= a_1g_1\cdot a_2e_1\cdot b_1g_3\cdot c_1e_3\cdot d_2g_4\cdot e_2g_2\cdot f_1f_3,\\
S_{4\ 7} &= a_1g_1\cdot a_2e_1\cdot b_1g_3\cdot c_1e_3\cdot d_2g_4\cdot e_2g_2\cdot f_2\alpha\,,\\
S_{4\ 8} &= a_1f_2\cdot a_2d_1\cdot b_2\alpha_1\cdot c_1g_3\cdot d_2f_1\cdot d_3g_1\cdot e_1\alpha_2,\\
S_{4\ 9} &= a_1g_1\cdot a_2e_2\cdot b_1f_3\cdot c_2\alpha_1\cdot d_1\alpha_2\cdot e_1g_2\cdot f_1g_3,\\
S_{4\ 10} &= a_1f_2\cdot a_2f_1\cdot b_1g_2\cdot b_2g_1\cdot d_1d_2\cdot d_3\alpha_1\cdot g_4\alpha_2,\\
S_{4\ 11} &= a_1g_2\cdot a_2g_1\cdot b_1f_2\cdot b_2f_1\cdot e_1e_2\cdot e_3\alpha_1\cdot g_4\alpha_2,\\
S_{4\ 12} &= a_1g_2\cdot a_2e_2\cdot b_1\alpha_1\cdot c_1\alpha_2\cdot d_1f_3\cdot e_1g_1\cdot f_2g_4,\\
S_{4\ 13} &= a_1g_2\cdot a_2e_1\cdot b_1g_4\cdot c_2d_3\cdot d_2g_3\cdot e_2g_1\cdot f_2\alpha\,.
\end{aligned}
$$

Cases (c) and (e) permit also

$$S_{4\ 14}=a_1g_2\cdot a_2e_1\cdot b_2\alpha_1\cdot c_1f_3\cdot d_1\alpha_2\cdot e_2g_1\cdot e_3f_1,$$

transformed by σ into $a_1g_2\cdot b_2f_1\cdot c_1\alpha_1\cdot c_2d_3\cdot d_1f_3\cdot d_2g_3\cdot e_3\alpha_2$, which with s_1 generates an octic group; and

$$
\begin{aligned}
S_{4\ 15} &= \sigma t_3 & S_{4\ 10}\,t_3\sigma &= a_1f_2\cdot a_2d_2\cdot b_1g_4\cdot c_1\alpha_1\cdot d_1f_1\cdot e_1g_3\cdot g_2\alpha_2,\\
S_{4\ 16} &= \sigma t_3 & S_{4\ 1}\,t_3\sigma &= a_1f_1\cdot a_2d_2\cdot b_1g_3\cdot c_2\alpha\cdot d_1f_2\cdot e_1g_4\cdot f_3g_1,\\
S_{4\ 17} &= \sigma & S_{4\ 2}\,\sigma &= a_1f_2\cdot a_2d_2\cdot b_2d_3\cdot c_2f_3\cdot d_1f_1\cdot e_2e_3\cdot g_1\alpha\,,\\
S_{4\ 18} &= \sigma & S_{4\ 4}\,\sigma &= a_1g_2\cdot a_2e_2\cdot b_2e_3\cdot c_2g_3\cdot d_2d_3\cdot e_1g_1\cdot f_1\alpha\,,\\
S_{4\ 19} &= s_1s_3 & S_{4\ 5}\,s_3s_1 &= a_1f_2\cdot a_2d_1\cdot b_1\alpha\cdot c_2e_3\cdot d_2f_1\cdot e_2f_3\cdot g_2g_4.
\end{aligned}
$$

The following relations reduce the number of the substitutions we shall have to test for s_4:

$$
\begin{aligned}
& s_1s_3t_1S_{4\ 1}\ t_2S_{4\ 1}\ t_1s_3s_1=\sigma S_{4\ 4}\sigma,\\
& t_3S_{4\ 8}\ t_2S_{4\ 8}\ t_3=S_{4\ 1},\\
& t_3S_{4\ 3}\ t_2S_{4\ 3}\ t_3=S_{4\ 4},\\
& s_3s_1s_2S_{4\ 9}\ t_3S_{4\ 9}\ s_2S_{4\ 9}\ t_3S_{4\ 9}\ s_2s_1s_3=S_{4\ 6},\\
& t_1S_{4\ 11}\ s_2S_{4\ 11}\ t_1=S_{4\ 1},\\
& s_3S_{4\ 12}\ t_3S_{4\ 12}\ s_3=\sigma S_{4\ 2}\,\sigma,\\
& S_{4\ 4}\ s_3S_{4\ 4}=S_{4\ 6},\\
& S_{4\ 10}\ t_2S_{4\ 10}\ s_2S_{4\ 10}\ t_2S_{4\ 10}=S_{4\ 5},
\end{aligned}
$$

Thus if $S_{4\ 6}$ were fully discussed, $S_{4\ 4}$, $S_{4\ 8}$, $S_{4\ 11}$, $S_{4\ 1}$, $S_{4\ 9}$, and $S_{4\ 3}$ would require no further attention, as whatever primitive groups they might lead to would be revealed in following up completely the implications of $S_{4\ 6}$. The substitutions then which we shall use for s_4 are $S_{4\ 7}$, $S_{4\ 6}$, $S_{4\ 2}$, $S_{4\ 5}$, and $S_{4\ 13}$.

15. No primitive group of the sort under discussion contains $H_4 = \{H_3, S_{4\,7}\}$. For since H_4 is invariant under the substitutions of $\{t_3, t_4\}$, where $t_4 = t_1 S_{4\,7}\, t_1 = b_1e_2 \cdot b_2e_1 \cdot c_1g_2 \cdot c_2g_1 \cdot d_3g_4 \cdot e_3g_3 \cdot f_3f_4$, it is sufficient to put $s_5 = (a_1f_1) \cdots$ or $(a_1f_2) \cdots$. Now when in s_5 a_1 is replaced by f_1 or f_2, s_5 can be so taken as to come under one of the cases (a), (b), $\cdots$ of § 14. But none of the nine substitutions available are consistent with $S_{4\,7}$.

16. Let $S_{4\,6}$ be our s_4. To the six substitutions of order 2 and degree 14 in H_3 we add

$$\begin{aligned}
s_4 &= a_1g_1 \cdot a_2e_1 \cdot b_1g_3 \cdot c_1e_3 \cdot d_2g_4 \cdot e_2g_2 \cdot f_1f_3,\\
t_4 &= b_1e_2 \cdot b_2e_1 \cdot c_1g_2 \cdot c_2g_1 \cdot d_3g_4 \cdot e_3g_3 \cdot f_1f_2,\\
t_5 &= a_1e_2 \cdot a_2g_2 \cdot b_2g_3 \cdot c_2e_3 \cdot d_1g_4 \cdot e_1g_1 \cdot f_2f_3,\\
t_6 &= a_1d_2 \cdot a_2d_1 \cdot c_1g_1 \cdot c_2g_2 \cdot d_3g_3 \cdot e_3g_4 \cdot f_1f_2,\\
t_7 &= a_2g_3 \cdot b_1d_2 \cdot b_2g_2 \cdot c_1d_3 \cdot d_1g_1 \cdot e_1g_4 \cdot f_2f_3,\\
t_8 &= a_1g_3 \cdot b_1g_1 \cdot b_2d_1 \cdot c_2d_3 \cdot d_2g_2 \cdot e_2g_4 \cdot f_1f_3.
\end{aligned}$$

This group H_4 is transformed into itself by σ (of § 14) and by $\tau = a_2b_1 \cdot b_2c_1 \cdot d_1g_1 \cdot d_2g_3 \cdot d_3g_2 \cdot e_2e_3 \cdot f_2f_3$.

The group H_3 has the invariant subgroup

$$\begin{aligned}
1,\quad s_1s_3 &= a_1d_2 \cdot a_2d_1 \cdot b_1e_2 \cdot b_2e_1 \cdot c_1c_2 \cdot d_3e_3 \cdot g_1g_2 \cdot g_3g_4,\\
s_2t_3 &= a_1b_1 \cdot a_2e_1 \cdot b_2d_1 \cdot c_1e_3 \cdot c_2d_3 \cdot d_2e_2 \cdot g_1g_3 \cdot g_2g_4,\\
t_1t_2 &= a_1e_2 \cdot a_2b_2 \cdot b_1d_2 \cdot c_1d_3 \cdot c_2e_3 \cdot d_1e_1 \cdot g_1g_4 \cdot g_2g_3.
\end{aligned}$$

This axial group is transformed into itself by

$$s_1s_2 = a_1c_1b_2 \cdot a_2b_1c_2 \cdot d_1d_2d_3 \cdot e_1e_2e_3 \cdot f_1f_2f_3 \cdot g_1g_2g_3,$$

and s_1 transforms the tetraedral group $\{s_1s_3, s_2t_3, s_1s_2\}$ into itself and thereby generates the group $\{s_1s_3, s_2t_3, s_1s_2, s_1\} = \{s_1, s_2, s_3\} = H_3$, of order 24. The 16 letters of the larger constituent of H_4 fall into four systems of imprimitively of four letters each in just three ways: $a_1a_2e_3g_3$, $d_1d_2d_3g_4$, $b_1c_1e_1g_1$, $b_2c_2e_2g_2$; $a_1b_1d_2e_2$, $a_2b_2d_1e_1$, $c_1c_2d_3e_3$, $g_1g_2g_3g_4$; or $a_1c_2d_1g_1$, $a_2c_1d_2g_2$, $b_1b_2d_3g_3$, $e_1e_2e_3g_4$. There are no other systems of imprimitivity in this constituent, not even systems of two letters each. Now

$$\begin{aligned}
s_2s_4 &= a_1g_3 \cdot a_2e_3 \cdot b_1g_1 \cdot b_2c_2 \cdot c_1e_1 \cdot d_1d_3 \cdot e_2g_2 \cdot d_2g_4,\\
s_1t_4 &= a_1a_2 \cdot b_1e_1 \cdot b_2e_2 \cdot c_1g_1 \cdot c_2g_2 \cdot d_1d_2 \cdot d_3g_4 \cdot e_3g_3,
\end{aligned}$$

and the transitive elementary group of order 16

$$\{s_1s_3,\ s_2t_3,\ s_2s_4,\ s_1t_4\}$$

is invariant in H_4. By the adjunction of s_1s_2, a group of order 48 is generated. The group $\{s_1s_3, s_2t_3, s_2s_4, s_1t_4, s_1s_2, s_1\} = H_4$ and is of order 96. There can not be more than $3 \cdot 16/4$ substitutions of degree 12 and order 2 in the transi-

tive constituent of order 96. The 12 substitutions $s_1, s_2, \cdots, t_8$ are conjugate in H_4. To extend this H_4 we have only to consider the single type of substitution $(a_1f_1)\cdots$, on account of t_3 and t_4, which fix a_1. Nor need we go outside the cases (a), (b), $\cdots$ of § 14. Since $\tau S_{4\,5}\tau = S_{4\,1}$ it is sufficient to use $S_{4\,1}$ for s_5. It will be convenient however to put

$$s_5 = t_6 S_{4\,1}\, t_6 = a_1a_2\cdot b_1c_1\cdot b_2c_2\cdot d_1f_1\cdot d_2f_2\cdot d_3f_3\cdot f_4g_4.$$

The group $\{H_3, s_5\}$ has a symmetric-5 constituent in the letters $g_1g_2g_3g_4f_4$. In the other constituent of degree 15 there is an octic subgroup $\{s_1, s_2t_3\}$ which fixes f_3. This includes $s_2t_3s_1s_2t_3 = s_3$, and $\{s_1, s_2t_3, s_5\}$ includes $(s_5s_2t_3)^3 = s_2^3(s_5t_3)^3 = s_2$, so that $\{s_1, s_2, s_3, s_5\}$ coincides with $\{s_1, s_2t_3, s_5\}$. If now the symmetric group of order 120 is represented as the group according to which its 15 octic subgroups are permuted, we have the above constituent of degree 15. Let the 15 octic groups in (12345) *all* be characterized by an operator of order 4:

$$\begin{gathered} a_1 = 1254,\ a_2 = 1245,\ b_1 = 2345,\ b_2 = 1345,\ c_1 = 2354, \\ c_2 = 1354,\ d_1 = 1253,\ d_2 = 1235,\ d_3 = 1325,\ e_1 = 2435, \\ e_2 = 1435,\ e_3 = 1425,\ f_1 = 1243,\ f_2 = 1432,\ f_3 = 1324; \end{gathered}$$

then 12, 13·24, and 45 give rise to

$$a_1a_2\cdot b_1b_2\cdot c_1c_2\cdot d_1d_2\cdot e_1e_2\cdot f_1f_2,$$
$$a_1b_1\cdot a_2e_1\cdot b_2d_1\cdot c_1e_3\cdot c_2d_3\cdot d_2e_2,$$

and

$$a_1a_2\cdot b_1c_1\cdot b_2c_2\cdot d_1f_1\cdot d_2f_2\cdot d_3f_3,$$

respectively. Hence $\{s_1, s_2, s_3, s_5\}$ is exactly of order 120 and is a simple isomorphism between its two constituents. We next note that H_5 has an invariant subgroup of order 16 that leaves every system of imprimitivity of four letters fixed. It is generated by

$$\begin{aligned} s_1t_4 &= a_1a_2\cdot b_1e_1\cdot b_2e_2\cdot c_1g_1\cdot c_2g_2\cdot d_1d_2\cdot e_3g_3\cdot d_3g_4, \\ s_2s_4 &= a_1g_3\cdot a_2e_3\cdot b_1g_1\cdot b_2c_2\cdot c_1e_1\cdot d_1d_3\cdot d_2g_4\cdot e_2g_2, \\ s_5s_1t_4s_5 &= a_1a_2\cdot b_1g_1\cdot b_2g_2\cdot c_1e_1\cdot c_2e_2\cdot e_3g_3\cdot f_1f_2\cdot f_3f_4, \\ s_5s_2s_4s_5 &= a_1e_3\cdot a_2g_3\cdot b_1e_1\cdot b_2c_2\cdot c_1g_1\cdot e_2g_2\cdot f_1f_3\cdot f_2f_4. \end{aligned}$$

The 15 substitutions of order 2 are similar and with the identity constitute an Abelian group. This group has no substitution other than the identity in common with $\{s_1, s_2t_3, s_5\}$ and therefore the two groups together generate a group of order 1920. This clearly includes s_2, s_3, and s_4. H_4, of order 96, is its subgroup that leaves one letter fixed. The only systems of imprimitivity permitted by H_5 are $a_1a_2e_3g_3$, $d_1d_2d_3g_4$, $b_1c_1e_1g_1$, $b_2c_2e_2g_2$, and $f_1f_2f_3f_4$, so that if H_5 is a subgroup of a primitive group of higher degree, it is in a multiply transitive group of degree 21. It is known that there is no primitive group of degree 20 and class 14.

17. There should now be sought a substitution $s_6 = (\alpha f_4)\cdots$ which with H_5 will generate a doubly transitive group of degree 21. Any one of the substitutions $s_1, s_2, \cdots, t_8$ generates with s_6 an Abelian or an octic group. Since s_6 certainly displaces one of the 16 letters $a_1, a_2, \cdots, g_4$, it may be assumed that g_4 is in s_6. Because $\{s_5, s_6 = (\alpha f_4)(g_4 -)\cdots\}$ is octic, the letter that follows g_4 in s_6 is e_1, e_2, e_3, g_1 g_2, or g_3, and after transformation, if necessary, by substitutions of H_2 we may make it e_3 or g_3. But H_5 is transformed into itself by $a_1a_2\cdot b_1c_1\cdot b_2c_2\cdot e_1g_1\cdot e_2g_2\cdot e_3g_3$, and therefore $s_6 = (\alpha f_4)(g_4e_3)\cdots$ if we like. Suppose then that s_4s_6 is of order 4; s_6 is commutative with s_1 and t_6, and in consequence fixes the 8 letters $c_1c_2d_1d_2a_1a_2g_1g_2$. Then s_4s_6 is of order 2 and

$$s_6 = a_1g_1\cdot a_2g_2\cdot c_1d_2\cdot c_2d_1\cdot e_1e_2\cdot e_3g_4\cdot f_4\alpha.$$

It will presently be shown that H_6 exists; that is, that H_5 is the subgroup of H_6 that leaves one letter fixed. We proceed on the assumption that H_6 exists and is of order 21·20·96.

Each subgroup of H_6 that leaves two letters fixed, as H_4, has one and only one regular subgroup of degree and order 16. Then there are just 21 of these subgroups of order 16 in H_6 and each is invariant in a subgroup of order 1920, one constituent of which is the symmetric group of degree 5. Since we have laid down the condition that G contains no substitution of degree 15 and no substitution of degree 14 and order 7, the largest subgroup of G in which the regular subgroup of degree 16, Abelian of type (1, 1, 1, 1), is invariant has no substitutions other than the 16 of the regular group on the 16 letters only of the latter group. Since the constituent in the other five letters is already a symmetric group, and since the subgroup of G that leaves two letters fixed can have but one regular elementary subgroup of order 16, G and H_6 are identical.

18. Now we seek to find a triply transitive group of degree 22 of which H_6 is a subgroup. Obviously s_7 is not $(\beta\alpha)(f_4g_4)\cdots$, because of s_6. Hence H_6 is not a subgroup of a quadruply transitive group with the present limitations. Since H_4 has a transitive constituent of degree 16, we can, after proper transformation, put $s_7 = (\beta\alpha)(f_4 -)(g_4 -)\cdots$. Since $\{s_6, s_7\}$ is octic, the letter by which s_7 replaces f_4 is fixed by s_6 and since we may use for s_7 any one of its transforms by substitutions of H_2, it is enough to consider $(\beta\alpha)(f_4f_3)\cdots$ only. We find

$$s_7 = \beta\alpha\cdot f_4f_3\cdot g_4d_3\cdot e_3g_3\cdot f_1f_2\cdot d_1d_2\cdot a_1a_2.$$

There exists a quintuply transitive group of degree 24 and of class 16,

discovered by Mathieu * and our groups H_6 and H_7 are concealed in it as transitive constituents of certain intransitive subgroups. In fact H_7 is the triply transitive constituent of that largest subgroup of Mathieu's group in which the subgroup that leaves two letters fixed is invariant. The generating substitutions of the quintuply transitive group are

$$\begin{aligned}
Z_1 &= (4, 19)(15, 12)(16, 7)(10, 8)(18, 9)(6, 13)(11, 20)(21, 22),\\
Z_2 &= (4, 16)(19, 7)(15, 10)(12, 8)(18, 11)(9, 20)(6, 21)(13, 22),\\
Z_3 &= (4, 18)(19, 9)(15, 6)(12, 13)(16, 11)(7, 20)(10, 21)(8, 22),\\
Z_4 &= (4, 12)(19, 15)(16, 8)(7, 10)(18, 13)(9, 6)(11, 22)(20, 21),\\
Y &= (3, 14, 17)\ (7, 19, 21)\ (13, 15, 12)\ (4, 10, 11)\ (9, 8, 16)\\
&\qquad (20, 18, 22),\\
U &= (2, 16, 9, 6, 8)\ (4, 3, 12, 13, 18)\ (10, 11, 22, 7, 17)\\
&\qquad (20, 15, 14, 19, 21),\\
B &= (1, 2, 4, 8, 16, 9, 18, 13, 3, 6, 12)\ (5, 10, 20, 17, 11, 22, 21, 19,\\
&\qquad 15, 7, 14),\\
A &= (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19,\\
&\qquad 20, 21, 22),\\
X &= (0, \infty)\ (1, 22)\ (2, 11)\ (3, 15)\ (4, 17)\ (5, 9)\ (6, 19)\ (7, 13)\\
&\qquad (8, 20)\ (10, 16)\ (12, 21)\ (14, 18).
\end{aligned}$$

If now the substitutions $A^{-2}Z_1A^2$, $Y^{-1}A^{-2}Z_1A^2Y$, $U^{-1}Y^{-1}A^{-2}Z_1A^2YU$, $U^{-1}Z_1UY^{-1}A^{-2}Z_1A^2YU^{-1}Z_1U$, $A^{-3}Z_4A^3$, $Z_1Z_4A^{-19}BZ_1B^{-1}A^{19}Z_4Z_1$, $(B^{-2}XA^{-2}Z_1A^2XB^2Z_3)^{-1}U^{-1}Z_1Z_2Z_3Z_4A^{-4}Z_2Z_3A^4Z_1Z_2Z_3Z_4U(B^{-2}XA^{-2}Z_1A^2XB^2Z_3)$, are transformed by $2g_4 \cdot 3d_3 \cdot 4e_3 \cdot 5\alpha \cdot 6g_1 \cdot 7g_3 \cdot 8a_1 \cdot 9e_2 \cdot 10f_2 \cdot 11c_2 \cdot 12f_1 \cdot 13b_2 \cdot 14d_2 \cdot 15a_2 \cdot 16f_3 \cdot 17d_1 \cdot 18e_1 \cdot 19f_4 \cdot 20c_1 \cdot 21g_2 \cdot 22b_1 \cdot \infty\beta$, they go into s_1, s_2, s_3, s_4, s_5, s_6, s_7, respectively, multiplied by the transposition $(0, 1)$.

19. With H_3 as before, let $S_{4\ 2}$ be our s_4:

$$\begin{aligned}
s_4 &= a_1f_2 \cdot a_2f_1 \cdot c_1e_1 \cdot c_2e_2 \cdot d_1d_2 \cdot e_3f_3 \cdot g_3g_5,\\
s_3s_4s_3 = t_4 &= a_1a_2 \cdot b_1c_1 \cdot b_2c_2 \cdot d_1f_1 \cdot d_2f_2 \cdot d_3f_3 \cdot g_4g_5,\\
s_1t_4 &= b_1c_2 \cdot b_2c_1 \cdot d_1f_2 \cdot d_2f_1 \cdot d_3f_3 \cdot e_1e_2 \cdot g_1g_2 \cdot g_4g_5.
\end{aligned}$$

For s_5 we have to try only $(a_1g_1)\cdots$ and $(a_1g_3)\cdots$. If $s_5 = (a_1g_3)(b_1)(g_1)\cdots$, $s_5s_2s_5 = (a_1g_1)\cdots$. If $s_5 = (a_1g_1)(a_2)(g_2)\cdots$, $s_1t_4s_5s_1s_5s_1t_4 = (a_1g_1)(a_2g_2)\cdots$. If $s_5 = (a_1g_1)(b_1)(g_3)\cdots$, $s_5s_2s_5 = (a_1g_3)(b_1g_1)\cdots$. Then s_5 is $S_{4\ 3}$, $S_{4\ 9}$, $(a_1g_3)(b_1g_1)\cdots$, or $(a_1g_3)(b_1 —)(g_1 —)\cdots$. But $S_{4\ 3}$ and $S_{4\ 9}$ depend (§ 14) on $S_{4\ 6}$. If $s_5 = (a_1g_3)(b_1g_1)\cdots$, we find at once the unique substitution

$$s_5 = a_1g_3 \cdot b_1g_1 \cdot b_2d_1 \cdot c_2d_3 \cdot d_2g_2 \cdot e_2g_4 \cdot f_1f_3 = s_1s_3S_{4\ 6}\,s_3s_1.$$

Unless g_1 in $s_5 = (a_1g_3)(b_1 —)(g_1 —)\cdots$ is followed by c_1 or e_1, s_5 can be

* Mathieu, *Liouville's Journal*, Ser. 2, Vol. 18 (1873), p. 25.

transformed by substitutions of H_4 into $(a_1g_1)\cdots$. Both letters are impossible.

20. Let $S_{4\,5}$ be s_4:

$$\begin{aligned} s_4 &= a_1f_1\cdot b_1f_3\cdot a_2d_1\cdot c_1d_3\cdot d_2f_2\cdot e_2f_4\cdot g_1g_3, \\ s_4t_1s_4 = t_4 &= b_1d_2\cdot b_2d_1\cdot c_1f_2\cdot c_2f_1\cdot d_3f_3\cdot e_3f_4\cdot g_1g_2, \\ t_2s_4t_2 = t_5 &= a_2e_1\cdot b_1d_2\cdot c_1f_3\cdot c_2f_4\cdot d_3f_2\cdot e_3f_1\cdot g_3g_4. \end{aligned}$$

Only $(a_1g_1)\cdots$ need be used for s_5. If $s_5=(a_1g_1)(b_1)(g_3)\cdots$, $t_5s_1s_3t_4s_5s_2s_5t_4s_1s_3t_5=(a_1g_1)(b_1g_3)\cdots$. If $s_5=(a_1g_1)(a_2)(g_2)\cdots$, $t_4s_5s_1s_5t_4=(a_1g_1)(a_2g_2)\cdots$. Thus we are restricted to the five cases of § 14 but none of those substitutions give us anything new.

21. Let $S_{4\,13}$ be our s_4.

$$\begin{aligned} s_4 &= a_1g_2\cdot a_2e_1\cdot b_1g_4\cdot c_2d_3\cdot d_2g_3\cdot e_2g_1\cdot f_2f_4, \\ s_3s_4s_3 = t_4 &= a_2g_4\cdot b_1d_2\cdot b_2g_1\cdot c_2e_3\cdot d_1g_2\cdot e_1g_3\cdot f_1f_4. \end{aligned}$$

Therefore we have only to try two types $(a_1\,f_1)\cdots$ and $(a_1f_2)\cdots$ for s_5. All the substitutions s_5 come under cases (a), (b), $\cdots$ of § 14, and they have all been discussed.

22. Referring to § 13, we take up the study of the other group H_3, of which the substitutions of order 2 and degree 14 are s_1, s_2, t_1,

$$\begin{aligned} s_3 &= a_1d_1\cdot a_2d_2\cdot b_1e_1\cdot b_2e_2\cdot d_3e_3\cdot f_3f_4\cdot g_3g_4, \\ t_2 &= a_1e_3\cdot b_1d_3\cdot c_1d_2\cdot c_2e_2\cdot d_1e_1\cdot f_1f_4\cdot g_1g_4, \\ t_3 &= a_2e_3\cdot b_2d_3\cdot c_1e_1\cdot c_2d_1\cdot d_2e_2\cdot f_2f_4\cdot g_2g_4. \end{aligned}$$

This group H_3 is invariant under $\sigma=a_2c_2\cdot b_2c_1\cdot d_1e_3\cdot d_2e_2\cdot d_3e_1\cdot f_1f_3\cdot g_1g_3$ and $f_1g_1\cdot f_2g_2\cdot f_3g_3\cdot f_4g_4$. Then s_4 is either $(a_1g_1)\cdots$ or $(a_1g_2)\cdots$. If we use the transformation by σ as in § 14, we have the same five cases as there and by the same brief process find

$$s_4 = a_1g_1\cdot a_2g_2\cdot b_1f_1\cdot b_2f_2\cdot d_3f_4\cdot e_3g_4\cdot f_3g_3.$$

It is true that two other substitutions seem possible with the present H_3; they are

$$\begin{aligned} s_4' &= a_1g_1\cdot a_2e_1\cdot b_1g_3\cdot c_1e_3\cdot d_2g_4\cdot e_2g_2\cdot f_1f_3, \\ s_4'' &= a_1g_1\cdot a_2e_1\cdot b_1g_3\cdot c_1e_3\cdot d_2g_4\cdot e_2g_2\cdot f_2f_4. \end{aligned}$$

But $a_2b_1\cdot b_2c_1\cdot d_1g_1\cdot d_2g_3\cdot d_3g_2\cdot e_2e_3\cdot f_2f_3$ transforms $\{s_2,\ s_1,\ s_4'\}$ into $\{s_1,\ s_2,\ a_1d_1\cdot a_2d_2\cdot b_1e_1\cdot b_2e_2\cdot d_3e_3\cdot f_1f_2\cdot g_3g_4\}$, the H_3 of § 14, while the latter group is transformed into $\{t_3,\ t_2,\ s_4''\}$ by $a_1c_1d_1e_3g_1d_3\cdot a_2e_1g_4g_3b_1d_2\cdot b_2e_2g_2\cdot f_1f_4f_3$.

Now H_4 is a transitive group of degree 20. If it is a subgroup of a primitive group, that group is multiply transitive, and in it there is a substitution

$(a_1g_2)\cdots$ of order 2 and degree 14. We have just seen that there is not such a substitution under our five cases. Then if $s=(a_1g_2)\cdots$ exists, $\{s, s_1\}$ is octic and $ss_1s=s_4$. But ss_1s and s_1 have no cycle in common, so that $\{s, s_1\}$ can not be octic. The substitution $s=(a_1g_2)\cdots$ therefore does not exist.

This completes the discussion of D_5, the diedral group of order 6 with one regular constituent.

23. In the group D_0 there are four substitutions of degree 14:

$$\begin{aligned} s_1 &= a_1a_2\cdot b_1b_2\cdot c_1c_2\cdot d_1d_2\cdot e_1e_2\cdot f_1f_2\cdot g_1g_2,\\ s_2 &= a_1b_1\cdot c_1\alpha_1\cdot c_2\alpha_2\cdot d_1\beta_1\cdot d_2\beta_2\cdot e_1\gamma_1\cdot e_2\gamma_2,\\ t_1 &= a_1b_2\cdot a_2b_1\cdot \alpha_1\alpha_2\cdot \beta_1\beta_2\cdot \gamma_1\gamma_2\cdot f_1f_2\cdot g_1g_2,\\ t_2 &= a_2b_2\cdot c_1\alpha_2\cdot c_2\alpha_1\cdot d_1\beta_2\cdot d_2\beta_1\cdot e_1\gamma_2\cdot e_2\gamma_1. \end{aligned}$$

All the substitutions of $\{f_1f_2,\ g_1g_2,\ f_1g_1\cdot f_2g_2,\ c_1c_2\cdot\alpha_1\alpha_2,\ d_1d_2\cdot\beta_1\beta_2,\ e_1e_2\cdot\gamma_1\gamma_2,\ c_1d_1\cdot c_2d_2\cdot\alpha_1\beta_1\cdot\alpha_2\beta_2,\ c_1e_1\cdot c_2e_2\cdot\alpha_1\gamma_1\cdot\alpha_2\gamma_2\}$ transform D_0 into itself. Then s_3 is either $(a_1c_1)\cdots$ or $(a_1f_1)\cdots$.

Let $s_3=(a_1c_1)\cdots$. If $s_1s_3=s_3s_1$, $\{s_2, s_3\}$ is octic, $s_3=(a_1c_1)(a_2c_2)(b_1)(b_2)\cdots$, and $(t_1s_3)^3=1$. Therefore $s_3=(a_1c_1)(f_1 —)(f_2)\cdots$ is impossible. If $s_2s_3=s_3s_2$, $s_3=(a_1c_1)(b_1\alpha_1)(a_2)(c_2)(\alpha_2)\cdots$ and $\{t_2, s_3\}$ is of order 6. Then $s_3=(b_2\delta_1)\cdots$ and replaces d_2 or β_1 by a letter new to t_2 and in s_2. This is impossible.

Now $\{s_1, s_3\}$ and $\{s_2, s_3\}$ are both octic; $\{t_2, s_3\}$ is octic because s_3 fixes c_2 and α_1. Hence s_3 should replace α_2 by a letter fixed by t_2 and displaced by s_2, and this is impossible because s_3 fixes b_1.

Let $s_3=(a_1f_1)\cdots$. Neither s_1 nor t_1 can be commutative with s_3. The groups $\{s_1, s_3\}$ and $\{t_1, s_3\}$ are octic. Let $(s_2s_3)^3=1$. Now s_3 can be transformed into $(a_1f_1)(c_1x)(d_1y)(e_1z)(a_2)(b_1)(b_2)(f_2)(\alpha_1)(\beta_1)(\gamma_1)\cdots$, where x, y, z are new to s_2, so that we may put $x=g_1$ or δ_1. But $(a_1f_1)(c_1g_1)(g_2)\cdots$ is inconsistent with t_1. So too is $(a_1f_1)(c_1\delta_1)(d_1\delta_2)(e_1\delta_3)\cdots$. Let $(s_2s_3)^4=1$. Immediately, $s_3=(a_1f_1)(b_1g_1)(a_2)(b_2)(f_2)(g_2)\cdots$. If $\{s_1, s_3\}$ is D_2, s_3 fixes both letters of one of the three cycles (c_1c_2), (d_1d_2), (e_1e_2), and by proper transformation it can be made (e_1e_2). Thus $s_3=(a_1f_1)(b_1g_1)(c_1x_1)(c_2x_2)(d_1y_1)(d_2y_2)(z_1z_2)(a_2)(b_2)(e_1)(e_2)(f_2)(g_2)\cdots$, where $x_1, x_2,\cdots$ are new to s_1. If $x_1=\alpha_2$, β_1, or β_2, s_3 fixes c_2, d_1, or d_2. If $x_1=\gamma_1$, γ_2 is replaced by d_1, d_2, or c_2; that is, by d_1 or c_2. If by c_2, $s_3=(d_1\beta_1)(d_2\beta_2)\cdots$ or $(d_1\beta_2)(d_2\beta_1)\cdots$. And $s_3=(a_1f_1)(b_1g_1)(c_1\gamma_1)(d_1\gamma_2)(a_2)(b_2)(e_1)(e_2)(f_2)(g_2)(\alpha_1)(\alpha_2)(\beta_1)(\beta_2)\cdots$ is impossible. Then finally $s_3=(a_1f_1)(b_1g_1)(c_1\delta_1)(c_2\delta_2)(d_1\delta_3)(d_2\delta_4)\cdots$. But $\{s_2, s_3\}$ is octic. If $\{s_1, s_3\}$ is D_0', $s_3=(a_1f_1)(b_1g_1)(a_2)(b_2)(f_2)(g_2)\cdots$ unites two of the three cycles (c_1c_2), (d_1d_2), (e_1e_2). The two fixed letters can be made c_2 and d_2.

Therefore $s_3 = (a_1f_1)(b_1g_1)(c_1d_1)(e_1x_1)(e_2x_2)(a_2)(b_2)(f_2)(g_2)(\alpha_1)(\beta_1)\cdots$ and is commutative with t_2, replacing α_2 by β_2. The letters x_1 and x_2 must be new to s_1 and in t_2, so that $s_3 = (e_1\gamma_1)(e_2\gamma_2)\cdots$ or $(e_1\gamma_2)(e_2\gamma_1)\cdots$, which, however, would make it commutative with s_2.

The proof is complete that G does not include D_0 or D_0'

24. The group D_2 has the four substitutions:

$$\begin{aligned}
s_1 &= a_1a_2 \cdot b_1b_2 \cdot c_1c_2 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2 \cdot g_1g_2,\\
s_2 &= a_1b_1 \cdot c_1d_1 \cdot e_1\alpha_1 \cdot e_2\alpha_2 \cdot f_1\beta_1 \cdot f_2\beta_2 \cdot \gamma_1\gamma_2,\\
t_1 &= a_1b_2 \cdot a_2b_1 \cdot c_1d_2 \cdot c_2d_1 \cdot \alpha_1\alpha_2 \cdot \beta_1\beta_2 \cdot g_1g_2,\\
t_2 &= a_2b_2 \cdot c_2d_2 \cdot e_1\alpha_2 \cdot e_2\alpha_1 \cdot f_1\beta_2 \cdot f_2\beta_1 \cdot \gamma_1\gamma_2.
\end{aligned}$$

It is transformed into itself by $\{s_1(g_1g_2),\ s_2(\gamma_1\gamma_2),\ a_1b_1\cdot a_2b_2,\ c_1d_1\cdot c_2d_2,\ e_1e_2\cdot\alpha_1\alpha_2,\ f_1f_2\cdot\beta_1\beta_2,\ a_1c_1\cdot a_2c_2\cdot b_1d_1\cdot b_2d_2,\ e_1f_1\cdot e_2f_2\cdot\alpha_1\beta_1\cdot\alpha_2\beta_2,\ a_1e_1\cdot a_2\alpha_1\cdot b_1e_2\cdot b_2\alpha_2\cdot c_1f_1\cdot c_2\beta_1\cdot d_1f_2\cdot d_2\beta_2\cdot g_1\gamma_1\cdot g_2\gamma_2\}$. There are therefore but the three following types to try for s_3:

$$(g_1a_1)\cdots,\qquad (g_1\gamma_1)\cdots,\qquad (g_1e_1)\cdots.$$

Let $s_3 = (g_1a_1)\cdots$. It is commutative with neither s_1 nor t_1. Then it fixes b_2. If b_1 is also fixed, $s_3 = (a_1g_1)(\gamma_1x)(\gamma_2)\cdots$, where x is new to s_2, and is a letter of t_2. After transformation by $c_1d_1\cdot c_2d_2$, $s_3 = (a_1g_1)(\gamma_1c_2)\cdots$. Because of s_1 and s_2, $s_3 = (a_1g_1)(\gamma_1c_2)(d_1)\cdots$, which is impossible with t_1. Then $s_3 = (a_1g_1)(b_1x)(a_2)(g_2)(b_2)\cdots$, where x is new to s_2 and in s_1, and therefore can be made c_2. Because of t_1, d_1 is fixed, and then because of s_1, d_2 is also fixed. Thus $\{t_2, s_3\}$ is neither of our two possible types of non-Abelian diedral groups. The primitive group G contains no substitution of degree 14 that replaces $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ by g_1 or g_2; nor $e_1, e_2, f_1, f_2, \alpha_1, \alpha_2, \beta_1, \beta_2$ by γ_1 or γ_2.

25. Let $s_3 = (g_1\gamma_1)\cdots$. If s_3 fixes g_2 or γ_2, we can transform it into $(g_1\gamma_1)(g_2)\cdots$. Now $(s_1s_3)^3 = 1$, and after another transformation $s_3 = (g_1\gamma_1)(a_1 —)(g_2)(a_2)(b_2)\cdots$ and this means that $\{t_2, s_3\}$ is octic; therefore $s_3 = (g_1\gamma_1)(\gamma_2x)(g_2)(a_1 —)(a_2)(b_2)\cdots$. Here x is new to s_2 and t_2, and is a letter of s_1, an absurdity. If $s_3 = (g_1\gamma_1)(g_2x)(\gamma_2y)\cdots$, x is new to s_1 and t_1, and y is new to s_2 and t_2; that is, $x = \delta_1$ and $y = \delta_2$. Then s_3 is either $(g_1\gamma_1)(g_2\gamma_2)\cdots$ or $(g_1\gamma_1)(g_2\delta_1)(\gamma_2\delta_2)\cdots$, and the four diedral groups generated by s_3 and s_1, s_2, t_1, and t_2 are octic. One of the four letters a_1, a_2, b_1, and b_2 is certainly displaced by s_3, and by appropriate transformation it can be made a_1. If $s_3 = (a_1\delta_3)\cdots$ it displaces a_2, b_1, and b_2. Not all these letters can be followed by letters new to D_2. Then we can say that x in $s_3 = (a_1x)\cdots$ is a letter of D_2. Moreover it is not b_1, b_2, or a_2; but after

transformation can be c_1, c_2, or e_1. Now $s_3=(g_1\gamma_1)(a_1c_1)\cdots$ fixes a_2, c_2, b_1, d_1, b_2 and d_2 and is impossible with s_2. Let $s_3=(a_1c_2)(a_2)(c_1)(b_2)(d_1)\cdots$. It must replace b_1 by a letter new to s_2 and in s_1; that is, by d_2. It must omit both e_1 and e_2 or else both f_1 and f_2. If e_1 and e_2 are fixed by s_3, we may transform it by $e_1f_1\cdot e_2f_2\cdot\alpha_1\beta_1\cdot\alpha_2\beta_2$. Now $s_3=(a_1c_2)(b_1d_2)(a_2)(c_1)(b_2)(d_1)(f_1)(f_2)(e_1x)(e_2y)\cdots$, where x and y are new to s_1 and are in s_2. Neither e is followed by an α. We can transform by $f_1f_2\cdot\beta_1\beta_2$ if necessary, and write $s_3=(a_1c_2)(b_1d_2)(e_1\beta_1)(e_2\beta_2)(a_2)(c_1)(b_2)(d_1)(f_1)(f_2)(\alpha_1)(\alpha_2)\cdots$. Therefore two substitutions s_3 are determined:

$$s_{3\,1}=g_1\gamma_1\cdot g_2\gamma_2\cdot a_1c_2\cdot b_1d_2\cdot e_1\beta_1\cdot e_2\beta_2\cdot\delta_1\delta_2,$$
$$s_{3\,2}=g_1\gamma_1\cdot g_2\delta_1\cdot\gamma_2\delta_2\cdot a_1c_2\cdot b_1d_2\cdot e_1\beta_1\cdot e_2\beta_2.$$

Now let $s_3=(g_1\gamma_1)(a_1e_1)(a_2)(e_2)(b_1)(\alpha_1)(b_2x)(\alpha_2y)\cdots$. Here x is new to t_1 and is in t_2, while y is new to t_2 and is in t_1. Hence y is c_1 or d_1 and we can make it c_1. It follows that s_3 replaces c_2 by a letter new to s_1 and fixes d_2. Then x is f_1 or f_2. Since transformation by $f_1f_2\cdot\beta_1\beta_2$ is possible, $x=f_1$. Now s_3 must replace c_2 by a letter new to s_1 and in t_2, but not by β_2. Therefore

$$s_{3\,3}=g_1\gamma_1\cdot g_2\gamma_2\cdot a_1e_1\cdot b_2f_1\cdot c_1\alpha_2\cdot c_2\beta_1\cdot\delta_1\delta_2,$$
or
$$s_{3\,4}=g_1\gamma_1\cdot g_2\delta_1\cdot\gamma_2\delta_2\cdot a_1e_1\cdot b_2f_1\cdot c_1\alpha_2\cdot c_2\beta_1.$$

26. Let $s_3=(g_1e_1)\cdots$. First let s_1s_3 be of order 4. Then $s_3=(e_1g_1)(e_2)(g_2)(\beta_1\delta_1)(\alpha x)\cdots$, and x is in s_1 and new to t_1, that is, x is f_1 or f_2. But not both $\{s_2, s_3\}$ and $\{t_2, s_3\}$ are octic. Now we have $s_3=(g_1e_1)(g_2e_2)\cdots$ and since $\{t_1, s_3\}$ is octic it displaces an α or a β. In case $s_3=(g_1e_1)(g_2e_2)(\alpha_1x)(\alpha_2y)\cdots$, x and y are new to s_2 and to t_2. There must be a cycle in s_3 new to s_2. If $s_3=(g_1e_1)(g_2e_2)(\alpha_1\delta_1)(\alpha_2\delta_2)(\delta_3\delta_4)\cdots$, it displaces no other letter new to s_2, or to t_2, and therefore is impossible. Then the new cycle in s_3 displaces two of the four letters a_2, b_2, c_2, and d_2. Of these six cycles, a_2b_2 and c_2d_2 are in t_2 and the other four are conjugate. Then

$$s_3=(g_1e_1)(g_2e_2)(\alpha_1\delta_1)(\alpha_2\delta_2)(a_2c_2)(a_1c_1)(b_1)(b_2)(d_1)(d_2)(\beta_1)(\beta_2)\cdots,$$

or finally,

$$s_{3\,5}=g_1e_1\cdot g_2e_2\cdot a_1c_1\cdot a_2c_2\cdot f_1f_2\cdot\alpha_1\delta_1\cdot\alpha_2\delta_2.$$

Now let $s_3=(g_1e_1)(g_2e_2)(\beta_1 —)(\beta_2 —)(\gamma_1 —)(\alpha_1)(\alpha_2)(\gamma_2)\cdots$. The letters following β_1, β_2, and γ_1 are new to s_2 and t_2, and both the letters of the cycle (f_1f_2) are fixed. Because of s_2, s_3 must displace a_1 or b_1 and also c_1 or d_1. Since s_3 can be transformed by $a_1b_1\cdot a_2b_2$ and by $c_1d_1\cdot c_2d_2$,

$$s_{3\,6} = g_1e_1 \cdot g_2e_2 \cdot \beta_1\delta_1 \cdot \beta_2\delta_2 \cdot \gamma_1\delta_3 \cdot a_1a_2 \cdot c_1c_2,$$

or

$$s_{3\,7} = g_1e_1 \cdot g_2e_2 \cdot \beta_1\delta_1 \cdot \beta_2\delta_2 \cdot \gamma_1\delta_3 \cdot a_1c_2 \cdot a_2c_1.$$

27. It is now necessary to consider the groups generated by $s_{3\,i}$ and D_2, $i = 1, 2, \cdots, 7$. We recall that G contains no substitution of degree 14 that replaces g_1 or g_2 by a_1, a_2, b_1, b_2, c_1, c_2, d_1, or d_2; nor γ_1 or γ_2 by e_1, e_2, f_1, f_2, α_1, α_2, β_1, or β_2.

Consider $s_{3\,1}$. A substitution $s_4 = (g_1x)\cdots$, where x is $e_1, e_2, \cdots, \beta_2$, is transformed by $s_{3\,1}$ into $(\gamma_1y)\cdots$, where y is one of the same letters. Then uniquely $s_4 = (g_1\delta_1)\cdots$. Because of $s_{3\,1}$ and $t_3 = a_2c_1 \cdot b_2d_1 \cdot e_1\beta_2 \cdot e_2\beta_1 \cdot g_1\gamma_2 \cdot g_2\gamma_1 \cdot \delta_1\delta_2$, neither $(g_1\delta_1)(\gamma_1\delta_2)\cdots$ nor $(g_1\delta_1)(\gamma_2\delta_2)\cdots$ is possible. Then s_4 fixes γ_1, γ_2 and δ_2. If $s_4 = (g_1\delta_1)(g_2x)\cdots$, x is new to s_1 and to t_1, but is in $s_{3\,1}$, impossible. Now $s_4 = (g_1\delta_1)(g_2)(\gamma_1)(\gamma_2)\cdots$ and $(s_1s_4)^3 = 1$. We seek a substitution $s_5 = (g_1\epsilon)\cdots$, where ϵ, new to H_3, replaces in s_4 a letter of s_1. Now $t = s_{3\,1}\, s_4s_{3\,1} = (\gamma_1\delta_2)(x\epsilon)(g_1)(g_2)\cdots$ and $ts_5t = (g_1x)\cdots$, where x is a letter of D_2, not g_2, γ_1, or γ_2. This disposes of $s_{3\,1}$, and a fortiori of $s_{3\,2}$.

For $s_{3\,3}$ and $s_{3\,4}$ the eight substitutions s_4 which may replace g_1 by $e_1, e_2, \cdots, \beta_2$ reduce on transformation by substitutions of D_2 to $(g_1e_2)\cdots$ and $(g_1f_2)\cdots$. Since $s_{3\,3}s_4s_{3\,3}$ (and $s_{3\,4}s_4s_{3\,4}$ likewise) is $(\gamma_1e_2)\cdots$ or $(\gamma_1f_2)\cdots$, $s_{3\,4}$ goes out at once and s_4 in the case of $s_{3\,3}$ can only be $(g_1\delta_1)\cdots$. Just as in the preceding paragraph this, too, is impossible.

We next take up $s_{3\,5}$. There are four transitive constituents in H_3. But s_4 can replace g_1 only by the letters of the constituent $f_1f_2\beta_1\beta_2$. Since s_2 and $f_1f_2 \cdot \beta_1\beta_2$ transform $\{D_2, s_{3\,5}\}$ into itself, there is but one substitution $s_4 = (g_1f_1)\cdots$ to be considered. Now $s_1s_4 \neq s_4s_1$, because of $s_{3\,5}$. Since it was proved that $(g_1e_1)\cdots$ must be commutative with s_1, $(g_1f_1)\cdots$ must also be commutative with s_1, contradicting the last remark.

28. Only $s_{3\,6}$ and $s_{3\,7}$ remain. The arguments leading to the rejection of $s_{3\,6}$ will be seen to be equally valid for $s_{3\,7}$.

$$\begin{aligned}
s_{3\,6} &= g_1e_1 \cdot g_2e_2 \cdot \beta_1\delta_1 \cdot \beta_2\delta_2 \cdot \gamma_1\delta_3 \cdot a_1a_2 \cdot c_1c_2,\\
t_3 &= \alpha_1g_1 \cdot \alpha_2g_2 \cdot f_1\delta_1 \cdot f_2\delta_2 \cdot \gamma_2\delta_3 \cdot b_1a_2 \cdot d_1c_2,\\
t_4 &= e_1g_2 \cdot e_2g_1 \cdot \beta_2\delta_1 \cdot \beta_1\delta_2 \cdot \gamma_1\delta_3 \cdot b_2b_1 \cdot d_2d_1,\\
t_5 &= \alpha_2g_1 \cdot \alpha_1g_2 \cdot f_2\delta_1 \cdot f_1\delta_2 \cdot \gamma_2\delta_3 \cdot a_1b_2 \cdot c_1d_2.
\end{aligned}$$

The substitution $f_1f_2 \cdot \beta_1\beta_2 \cdot \delta_1\delta_2$ transforms H_3 into itself. There are then but the three substitutions, $(g_1\delta_3)\cdots$, $(g_1f_1)\cdots$, $(g_1\delta_1)\cdots$, to try for s_4. Let $s_4 = (g_1\delta_3)\cdots$: $s_{3\,6}\, s_4s_{3\,6} = (\gamma_1e_1)\cdots$, which has been shown to be impossible. Let $s_4 = (g_1f_1)\cdots$. It is commutative with s_1. Then $s_4 = (g_1f_1)(g_2f_2)(\alpha_1\delta_1)(\alpha_2\delta_2)\cdots$ if it is commutative with t_3 also. Since $s_{3\,6}\, s_4$ is of order 3,

s_4 displaces γ_1 or δ_3, not both. Like $s_{3\,6}$ itself two cycles of s_4 are on four of the eight letters $a_1, a_2, \cdots, d_2$. Therefore one cycle of s_4 is $(\gamma_1\epsilon)$, and s_5 can be nothing but $(g_1\epsilon)\cdots$. But $s_4s_5s_4=(f_1\gamma_1)\cdots$. Suppose that s_4 is not commutative with t_3. If $s_4=(g_1f_1)(g_2f_2)(e_1x)(e_2y)\cdots$, x and y are in s_1 and not in $s_{3\,6}$ or t_4, an impossibility. If $s_4=(g_1f_1)(g_2f_2)(\alpha_1)(\alpha_2)(e_1)(e_2)\cdots$, it fixes the letters of two cycles of s_2 and is not commutative with s_2. Let $s_4=(g_1\delta_1)\cdots$. It may be that $s_{3\,6}$ and s_4 are commutative. If so, $s_4=(g_1\delta_1)(e_1\beta\ )\cdots$. If s_4 displaces g_2, it does not replace it by e_2 (because of s_1) nor by a new letter ϵ. But $(g_1\delta_1)(g_2\delta_2)(e_1\beta_1)(e_2\beta_2)(\alpha_1)(\alpha_2)\cdots$ is impossible with s_2. If s_4 fixes g_2, it fixes also e_2, β_2, and δ_2, and s_1s_4 is of order 3. Now $s_4=(g_1\delta_1)(\alpha_1f_1)(e_1\beta_1)(a\text{---})(b\text{---})(c\text{---})(d\text{---})(g_2)(e_2)(f_2)(\alpha_2)(\beta_2)(\delta_2)\cdots$ if $t_3s_4=s_4t_3$, and

$$s_4=(g_1\delta_1)(e_1\beta_1)(\alpha_2f_2)(e_2)(g_2)(f_1)(\alpha_1)(\beta_2)(\delta_2)\cdots \text{ if } t_3s_4 \neq s_4t_3.$$

All the letters of the two sets $g_1g_2e_1e_2\alpha_1\alpha_2$ and $f_1f_2\beta_1\beta_2\delta_1\delta_2$ of H_3 are accounted for, so that H_4 is certainly intransitive and s_5 can replace g_1 only by a letter ϵ of $s_4=(x\epsilon)\cdots$, x being one of the eight letters $a_1a_2b_1b_2c_1c_2d_1d_2$. A substitution of D_2 (s' say) transforms $(\delta_1x)\cdots$ into $(\delta_1x')\cdots$, where x' is fixed by s_4. Then $s_4s's_4s_5s_4s's_4=(g_1x')\cdots$, known to be impossible. Thus s_4 is not commutative with $s_{3\,6}$, nor with its transforms by s_2 or $t_2:t_3$ and t_5. Hence finally $s_4=(g_1\delta_1)(e_2\beta_2)(e_1)(f_1)(f_2)\cdots$ or $(g_1\delta_1)(e_1)(e_2)(f_1)(f_2)\cdots$ and these too are impossible.

Hence no G contains D_2 without also containing D_5. *In what follows the only non-Abelian diedral group is D_4.*

29. There are 10 Abelian groups generated by two substitutions of degree 14.

Consider the group generated by

$$s_1 \text{ and } s_2=a_1b_1\cdot a_2b_2\cdot c_1d_1\cdot c_2d_2\cdot e_1f_1\cdot e_2f_2\cdot\alpha_1\alpha_2.$$

We may transform tnis group by $a_1a_2\cdot b_1b_2$ and by $a_1b_1\cdot a_2b_2$. Then only $(g_1a_1)\cdots$ and $(g_1\alpha_1)\cdots$ need be considered for s_3. Let $s_3=(g_1a_1)\cdots$. We have immediately $s_3=(g_1a_1)(g_2a_2)(b_1)(b_2)\cdots$, and since we may transform by $c_1d_1\cdot c_2d_2$ and by $e_1f_1\cdot e_2f_2$ if necessary, $s_3=(g_1a_1)(g_2a_2)(c_1x)(c_2y)(e_1z)(e_2w)(\alpha_1\text{---})(b_1)(b_2)(\alpha_2)(d_1)(d_2)(f_1)(f_2)$, where x, y, z and w should be letters of s_1. Let $s_3=(g_1\alpha_1)\cdots$. Uniquely $s_3=g_1\alpha_1\cdot a_1\delta_1\cdot b_2\delta_2\cdot c_1\delta_3\cdot d_2\delta_4\cdot e_1\delta_5\cdot f_2\delta_6$. Next $s_4=(g_1\delta_i)\cdots$ and $s_3s_4s_3=(\alpha_1x)\cdots$, where x is a letter of s_1, not g_1 or g_2. But this, as shown above, is not possible.

Consider s_1 and $s_2=a_1b_1\cdot a_2b_2\cdot c_1d_1\cdot c_2d_2\cdot e_1e_2\cdot\alpha_1\alpha_2\cdot\beta_1\beta_2$. Now s_3 is

$(e_1a_1)\cdots$ or $(e_1f_1)\cdots$. If we try the first, $s_3=(e_1a_1)(e_2a_2)\cdots$ by s_1 and $(e_1a_1)(e_2b_1)\cdots$ by s_2. If $s_3=(e_1f_1)(e_2f_2)\cdots$, $\{s_2, s_3\}$ is octic.

Consider s_1 and $s_2=a_1b_1\cdot a_2b_2\cdot c_1c_2\cdot\alpha_1\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2\cdot\delta_1\delta_2$,

$$s_1 \text{ and } s_2=a_1a_2\cdot\alpha_1\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2\cdot\delta_1\delta_2\cdot\epsilon_1\epsilon_2\cdot\zeta_1\zeta_2.$$

For s_3 both $(c_1a_1)\cdots$ and $(c_1d_1)\cdots$ are impossible.

30. Consider s_1 and $s_2=a_1b_1\cdot a_2b_2\cdot c_1c_2\cdot d_1d_2\cdot\alpha_1\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2$. Both $(c_1a_1)\cdots$ and $(c_1e_1)\cdots$ are impossible as before. Let $s_3=(c_1d_1)(c_2d_2)\cdots$. No s_4 replaces c_1 by one of the other 16 letters of our group D. Then s_3 has at least one new cycle $(\delta_1\delta_2)$. In fact s_4 is uniquely determined as $s_4=c_1\delta_1\cdot d_1\delta_2\cdot a_1\epsilon_1\cdot b_2\epsilon_2\cdot e_1\alpha_1\cdot f_1\beta_1\cdot g_1\gamma_1$. Since transformation by $a_1b_2\cdot a_2b_1\cdot\epsilon_1\epsilon_2$ is possible, s_5 is $(c_1\epsilon_1)\cdots$ or $(c_1\zeta_1)\cdots$, $(\zeta_1\zeta_2)$ being another cycle of s_3. If $s_5=(c_1\epsilon_1)\cdots$, $s_1s_5s_1=(c_2\epsilon_1)(c_1)\cdots$, and $s_4s_1s_5s_1s_4=(c_2a_1)\cdots$. We now have $s_5=(c_1\zeta_1)(d_1\zeta_2)(c_2)(d_2)\cdots$. Because the transitive constituent $c_1,\cdots$ is increased only by the addition of the two letters of a cycle of s_3 at this and each succeeding step, the substitutions $s_1, s_2, \cdots$ cannot in this case generate a transitive group.

Consider s_1 and $s_2=a_1a_2\cdot b_1b_2\cdot\alpha_1\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2\cdot\delta_1\delta_2\cdot\epsilon_1\epsilon_2$. The reasoning of the preceding paragraph is again applicable.

31. Consider s_1 and $s_2=a_1a_2\cdot b_1b_2\cdot c_1c_2\cdot\alpha_1\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2\cdot\delta_1\delta_2$. This is the last of the Abelian D groups in which s_1 and s_2 have a cycle in common. Therefore $s_3=(a_1b_1)(a_2b_2)(c_1)(c_2)(\epsilon_1\epsilon_2)\cdots$. To follow this we have $s_4=(a_1c_1)(a_2c_2)\cdots$ or $s_4=(a_1\epsilon_1)(b_1\epsilon_2)(c_1\kappa)\cdots$. If $s_4=(a_1c_1)(a_2c_2)\cdots$, $s_5=(a_1\epsilon_1)(b_1\epsilon_2)(c_1\epsilon_3)(a_2)(b_2)(c_2)\cdots$ or $(a_1\epsilon_2)(b_1\epsilon_1)(c_2\lambda)(\epsilon_3)(a_2)(b_2c_1)\cdots$. In either case the letters of the last four cycles of s_5 are letters of s_1 and s_2. Like s_5, s_6 will replace a_1 by a letter of a transitive constituent of $\{s_3, s_4\}$ of degree 3, and so on. But these steps will not lead us to a transitive group. If $s_4=(a_1\epsilon_1)(b_1\epsilon_2)(c_1\kappa)\cdots$, it can be completed as follows:

$s_4=a_1\epsilon_1\cdot b_1\epsilon_2\cdot c_1\kappa\cdot d_1\alpha_1\cdot e_1\beta_1\cdot e_2\gamma_1\cdot f_1\delta_1$. If $s_5=(a_1\kappa)\cdots$, $s_4s_1s_5s_1s_4=(a_2c_1)\cdots$,

already discussed. If $s_5=(a_1\zeta_1)(b_1\zeta_2)\cdots$, the last four cycles are again built up from the last four cycles of s_1 and s_2. The third cycle is $(c_1\lambda)$ or $(c_2\kappa)$. It is evident that we are not going to build up a transitive group by continuing with $s_6, s_7, \cdots$.

32. Consider s_1 and $s_2=a_1b_1\cdot a_2b_2\cdot c_1d_1\cdot c_2d_2\cdot\alpha_1\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2$. For s_3 we have $(e_1a_1)\cdots$, $(e_1f_1)\cdots$, or $(e_1\alpha_1)\cdots$. It is to be remembered that *no two substitutions of degree* 14 *have a common cycle.* In the first case, immediately and uniquely, $s_3=e_1a_1\cdot e_2a_2\cdot c_1f_1\cdot c_2f_2\cdot\alpha_1\alpha_3\cdot\beta_1\beta_3\cdot\gamma_1\gamma_3$. This case can

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be continued by either $(g_1 a_1) \cdots$ or $(g_1 \gamma_1) \cdots$. Since H_3 is transformed into itself by $a_1 a_2 \cdot b_1 b_2 \cdot e_1 e_2 \cdot$ and $c_1 c_2 \cdot d_1 d_2 \cdot f_1 f_2$, we get

$$s_{4\,1} = g_1 a_1 \cdot g_2 a_2 \cdot d_1 f_2 \cdot d_2 f_1 \cdot \alpha_1 \alpha_4 \cdot \beta_1 \beta_4 \cdot \gamma_1 \gamma_4,$$
$$s_{4\,2} = g_1 a_1 \cdot g_2 a_2 \cdot d_1 f_2 \cdot d_2 f_1 \cdot \alpha_1 \alpha_4 \cdot \beta_2 \gamma_3 \cdot \gamma_2 \beta_3,$$
$$s_{4\,3} = g_1 \gamma_1 \cdot e_1 \alpha_2 \cdot b_1 \alpha_3 \cdot f_1 \beta_2 \cdot d_1 \beta_3 \cdot a_2 \alpha_4 \cdot c_2 \beta_4.$$

Let us take up the three corresponding groups $H_{4\,1}$, $H_{4\,2}$ and $H_{4\,3}$ in turn. Because $c_1 c_2 \cdot d_1 d_2 \cdot f_1 f_2$ transforms $H_{4\,1}$ into itself, only $(g_1 c_1) \cdots$, $(g_1 \alpha_1) \cdots$, $(g_1 \alpha_4) \cdots$ need be tried for s_5. A short calculation shows that all three are impossible. With $H_{4\,2}$ there are four possibilities for s_5 : $(g_1 c_1) \cdots$, $(g_1 \alpha_1) \cdots$, $(g_1 \alpha_4) \cdots$ and $(g_1 \beta_1) \cdots$. The substitution s_5 can be completed only in the last case, $s_5 = g_1 \beta_1 \cdot a_2 \delta \cdot b_1 \gamma_3 \cdot e_1 \gamma_2 \cdot c_1 \alpha_4 \cdot d_2 \alpha_3 \cdot f_2 \alpha_2$. The transitive sets of H_5 are $c_1 c_2 d_1 d_2 f_1 f_2 \alpha_1 \alpha_2 \alpha_3 \alpha_4$ and one other, $\delta a_1 \cdots$. Only two forms of s_6 are to be tried, $(\delta c_1) \cdots$ and $(\delta \alpha_1) \cdots$; or, if we prefer, their transforms, $(g_1 \alpha_1) \cdots$ and $(g_1 \alpha_4) \cdots$. These two substitutions were inadmissible when proposed for s_5. Let us consider $H_{4\,3}$. To extend this group we have $(g_1 \alpha_1) \cdots$, $(g_1 \alpha_4) \cdots$, and $(g_1 a_1) \cdots$. Calculation shows that there are but two substitutions s_5, one of which is $s_5 = g_1 \alpha_1 \cdot \gamma_2 e_1 \cdot b_1 \gamma_3 \cdot a_2 \delta_1 \cdot c_1 \beta_4 \cdot f_2 \beta_2 \cdot d_2 \beta_3$. With this group H_5 only one substitution s_6 need be considered because H_5 has just two transitive sets, one of which is $c_1 c_2 d_1 d_2 f_1 f_2 \beta_1 \beta_2 \beta_3 \beta_4$. We may put $s_6 = (\delta_1 b_4) \cdots$ and this is at once seen to be impossible. The other $s_5 = g_1 a_1 \cdot g_2 a_2 \cdot \gamma_2 \alpha_3 \cdot \gamma_3 \alpha_2 \cdot f_1 d_2 \cdot f_2 d_1 \cdot \beta_1 \beta_4$ goes out with $s_{4\,2}$.

33. The second possibility for s_3, $(e_1 \alpha_1) \cdots$, is to be studied. Uniquely, $s_3 = e_1 \alpha_1 \cdot f_1 \beta_1 \cdot g_1 \gamma_1 \cdot a_1 \delta_1 \cdot b_2 \delta_2 \cdot c_1 \epsilon_1 \cdot d_2 \epsilon_2$. The group H_3 is on the five sets of letters: $a_1 a_2 b_1 b_2 \delta_1 \delta_2$, $c_1 c_2 d_1 d_2 \epsilon_1 \epsilon_2$, $e_1 e_2 \alpha_1 \alpha_2$, $f_1 f_2 \beta_1 \beta_2$, $g_1 g_2 \gamma_1 \gamma_2$. For s_4 we can have only $(e_1 f_1) \cdots$ or $(e_1 f_2) \cdots$. Now $(e_1 f_2) \cdots$ is impossible because it would have to replace g_1 or γ_1 by one of the letters $a_1, a_2, \cdots, d_2$. Then $s_4 = e_1 f_1 \cdot e_2 f_2 \cdot \alpha_1 \beta_1 \cdot \alpha_2 \beta_2 \cdot a_1 b_2 \cdot a_2 b_1 \cdot \delta_1 \delta_2$ or $s_4 = e_1 f_1 \cdot e_2 f_2 \cdot \alpha_1 \beta_1 \cdot \alpha_2 \beta_2 \cdot \zeta_1 \zeta_2 \cdot \eta_1 \eta_2 \cdot \theta_1 \theta_2$. The sets of H_4 are $e_1 e_2 f_1 f_2 \alpha_1 \alpha_2 \beta_1 \beta_2$, $a_1 a_2 b_1 b_2 \delta_1 \delta_2$, $g_1 g_2 \gamma_1 \gamma_2, \cdots$. Then $s_5 = (e_1 g_1) \cdots$, $(e_1 g_2) \cdots$, or, in case $s_4 = (\zeta_1 \zeta_2) \cdots$, $(e_1 \zeta_1) \cdots$. We need not have written down $(e_1 g_2) \cdots$ for since $(e_1 f_2) \cdots$ cannot be s_4, it cannot be s_5. Now $s_5 = (e_1 g_1)(e_2 g_2)(\alpha_1 \gamma_1)(\alpha_2 \gamma_2)(f_1) \cdots$. Considering only the first form of s_4, it is seen that s_5 must displace a_1 or b_2, and by means of $s_1 s_2$ this letter can be made a_1. But $s_5 = (e_1 g_1)(e_2 g_2)(\alpha_1 \gamma_1)(\alpha_2 \gamma_2)(a_1 c_1) \cdots$ or $(e_1 g_1)(e_2 g_2)(\alpha_1 \gamma_1)(\alpha_2 \gamma_2)(a_1 d_2) \cdots$ requires eight cycles. There remains only $s_4 = e_1 f_1 \cdot e_2 f_2 \cdot \alpha_1 \beta_1 \cdot \alpha_2 \beta_2 \cdot \zeta_1 \zeta_2 \cdot \eta_1 \eta_2 \cdot \theta_1 \theta_2$. Try $s_5 = e_1 g_1 \cdot e_2 g_2 \cdot \alpha_1 \gamma_1 \cdot \alpha_2 \gamma_2 \cdot \zeta_1 \zeta_3 \cdot \eta_1 \eta_3 \cdot \theta_1 \theta_3$. For s_6 we have only $(e_1 \zeta_1) \cdots$ and $(e_1 \zeta_2) \cdots$. If $s_6 = (e_1 \zeta_1)(f_1 \zeta_2)(g_1 \zeta_3)(ax) \cdots$, it fixes α_1, α_2, β_1, β_2, γ_1 and γ_2, while x should be in s_2 and not in s_1, an impossibility. If $s_6 = (e_1 \zeta_2)(f_1 \zeta_1) \cdots$, it fixes

g_1, α_1, α_2, β_1 and β_2; therefore it must replace γ_1 by a letter of one of the first four cycles of s_2, which takes us back to the case of $s_3 = (e_1 a_1)\cdots$. Finally try $s_5 = (e_1\zeta_1)(f_1\zeta_2)\cdots$. This is impossible with s_1 and s_2.

The third and last possibility for s_3 is to be studied: $s_3 = (e_1 f_1)(e_2 f_2)\cdots$. The substitutions $s_3, s_4, \cdots$ can replace e_1 only by g_1, g_2, or a letter δ new to H_2, and any one of these substitutions is commutative with s_2. Hence no one of them adds a new letter to $\alpha_1\alpha_2$, $\beta_1\beta_2$, or $\gamma_1\gamma_2$, and therefore the group $\{s_1, s_2, s_3, \cdots\}$ is not transitive.

This disposes of the group

$$\{s_1, s_2 = a_1b_1\cdot a_2b_2\cdot c_1d_1\cdot c_2d_2\cdot \alpha_1\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2\}.$$

Only two Abelian D groups, one of degree 24 and one of degree 28, and only one non-Abelian D group, of order 3 and degree 21, are left.

34. In the group D_4 there are the following three substitutions:

$$\begin{aligned} s_1 &= a_1a_2\cdot b_1b_2\cdot c_1c_2\cdot d_1d_2\cdot e_1e_2\cdot f_1f_2\cdot g_1g_2,\\ s_2 &= a_1a_3\cdot b_1b_3\cdot c_1c_3\cdot d_1d_3\cdot e_1e_3\cdot f_1f_3\cdot g_1g_3,\\ t &= a_2a_3\cdot b_2b_3\cdot c_2c_3\cdot d_2d_3\cdot e_2e_3\cdot f_2f_3\cdot g_2g_3.\end{aligned}$$

There is at least one substitution s_3, similar to s_1, non-commutative with at least two of the above three substitutions. We assume without loss of generality that s_3 is not commutative with s_1 and s_2. Since this substitution s_3 may or may not connect the transitive sets of D_4, two cases arise, as $\{t, s_3\}$ is of degree 24 or 28.

35. Case I:

$$s_3 = a_2b_3\cdot a_3b_2\cdot c_1c_4\cdot d_1d_4\cdot e_1e_4\cdot f_1f_4\cdot g_1g_4.$$

In H_3 are also $t_1 = a_1b_3\cdot a_3b_1\cdot c_2c_4\cdot d_2d_4\cdot e_2e_4\cdot f_2f_4\cdot g_2g_4$,

$$t_2 = a_1b_2\cdot a_2b_1\cdot c_3c_4\cdot d_3d_4\cdot e_3e_4\cdot f_3f_4\cdot g_3g_4.$$

We may extend H_3 by $s_4 = a_1c_1\cdot a_2c_2\cdot a_3c_3\cdot d_4d_5\cdot e_4e_5\cdot f_4f_5\cdot g_4g_5$ or by $a_1c_3\cdot a_3c_1\cdot b_2c_4\cdot d_2d_5\cdot e_2e_5\cdot f_2f_5\cdot g_2g_5$. These two substitutions are conjugate under $a_2b_2\cdot c_1c_3\cdot c_2c_4\cdot d_1d_3\cdot d_2d_4\cdot e_1e_3\cdot e_2e_4\cdot f_1f_3\cdot f_2f_4\cdot g_1g_3\cdot g_2g_4$, which transforms s_1, s_2, s_3 into t_2, s_2, s_3, respectively. Then we use the first only. In H_4 there are also the substitutions

$$\begin{aligned} t_3 &= a_1c_4\cdot b_3c_2\cdot b_2c_3\cdot d_1d_5\cdot e_1e_5\cdot f_1f_5\cdot g_1g_5,\\ t_4 &= b_3c_1\cdot a_2c_4\cdot b_1c_3\cdot d_2d_5\cdot e_2e_5\cdot f_2f_5\cdot g_2g_5,\\ t_5 &= b_2c_1\cdot b_1c_2\cdot a_3c_4\cdot d_3d_5\cdot e_3e_5\cdot f_3f_5\cdot g_3g_5.\end{aligned}$$

If s_5 replaces a_1 by d_3, $s_5 = a_1d_3\cdot a_3d_1\cdot b_2d_4\cdot c_2x\cdot e_2y\cdot f_2z\cdot g_2w$, where x is new to H_4, y, z and w are new to H_3, but displaced by s_4. Then $s_5 = a_1d_3\cdot a_3d_1\cdot b_2d_4\cdot c_2\alpha\cdot e_2f_5\cdot f_2e_5\cdot g_2$ —, an impossibility. Then

$$s_5 = a_1d_1 \cdot a_2d_2 \cdot a_3d_3 \cdot c_4d_5 \cdot e_4e_6 \cdot f_4f_6 \cdot g_4g_6.$$

This s_5 is unique because $(a_1d_4)\cdots$ and $(a_1d_2)\cdots$ can be transformed into $(a_1d_1)\cdots$ or $(a_1d_3)\cdots$ by a substitution of H_4 that fixes a_1. Next,

$$\begin{aligned} s_6 &= a_1e_1 \cdot a_2e_2 \cdot a_3e_3 \cdot c_4e_5 \cdot d_4e_6 \cdot f_4f_7 \cdot g_4g_7, \\ s_7 &= a_1f_1 \cdot a_2f_2 \cdot a_3f_3 \cdot c_4f_5 \cdot d_4f_6 \cdot e_4f_7 \cdot g_4g_8, \\ s_8 &= a_1g_1 \cdot a_2g_2 \cdot a_3g_3 \cdot c_4g_5 \cdot d_4g_6 \cdot e_4g_7 \cdot f_4g_8. \end{aligned}$$

The groups H_6, H_7, H_8 are unique. H_8 is simply isomorphic to the symmetric group of degree 9. In fact it is the group according to which that symmetric group permutes its 36 transpositions. Because of the impossibility of a substitution $(a_1d_3)\cdots$ of degree 14, H_8 is not a subgroup of a double transitive group of the same or higher degree. H_8 contains all the permissible substitutions similar to s_1 on the letters of H_8 only, except perhaps $(a_1b_1)\cdots$, $(a_1c_2)\cdots$ and $(a_1c_3)\cdots$. Now t_4 and t_5 reduce $(a_1c_2)\cdots$ and $(a_1c_3)\cdots$ to $(a_1b_1)\cdots$. This last substitution $(a_1b_1)(a_2b_2)(a_3b_3)\cdots$ fixes all the other letters of H_3, and therefore displaces d_5, e_5, f_5 and g_5, because of s_4. In the same way s_5 demands e_6, f_6, g_6, s_6 calls for f_7, g_7, and s_7 for g_8. Hence H_8 is invariant in any larger primitive group G, if one exists. But H_8 is a complete group.

36. Case II.

$$s_3 = a_1a_4 \cdot b_1b_4 \cdot c_1c_4 \cdot d_1d_4 \cdot e_1e_4 \cdot f_1f_4 \cdot g_1g_4.$$

The case in which s_i $(i = 3, 4, \cdots)$ is non-commutative with the substitutions of such a group as D_4 and connects two of its sets of transitivity has been fully considered. Then since we know that there is a substitution similar to s_1 which replaces a_1 by a letter of another constituent of H_i $(i = 3, 4 \cdots)$, we write down immediately and uniquely:

$$\begin{aligned} s_4 &= a_1b_1 \cdot a_2b_2 \cdot a_3b_3 \cdot a_4b_4 \cdot a_5b_5 \cdot a_6b_6 \cdot a_7b_7, \\ s_5 &= a_1c_1 \cdot a_2c_2 \cdot a_3c_3 \cdot a_4c_4 \cdot a_5c_5 \cdot a_6c_6 \cdot a_7c_7, \\ s_6 &= a_1d_1 \cdot a_2d_2 \cdot a_3d_3 \cdot a_4d_4 \cdot a_5d_5 \cdot a_6d_6 \cdot a_7d_7, \\ s_7 &= a_1e_1 \cdot a_2e_2 \cdot a_3e_3 \cdot a_4e_4 \cdot a_5e_5 \cdot a_6e_6 \cdot a_7e_7, \\ s_8 &= a_1f_1 \cdot a_2f_2 \cdot a_3f_3 \cdot a_4f_4 \cdot a_5f_5 \cdot a_6f_6 \cdot a_7f_7, \\ s_9 &= a_1g_1 \cdot a_2g_2 \cdot a_3g_3 \cdot a_4g_4 \cdot a_5g_5 \cdot a_6g_6 \cdot a_7g_7, \\ s_{10} &= a_1a_5 \cdot b_1b_5 \cdot c_1c_5 \cdot d_1d_5 \cdot e_1e_5 \cdot f_1f_5 \cdot g_1g_5, \\ s_{11} &= a_1a_6 \cdot b_1b_6 \cdot c_1c_6 \cdot d_1d_6 \cdot e_1e_6 \cdot f_1f_6 \cdot g_1g_6, \\ s_{12} &= a_1a_7 \cdot b_1b_7 \cdot c_1c_7 \cdot d_1d_7 \cdot e_1e_7 \cdot f_1f_7 \cdot g_1g_7. \end{aligned}$$

This imprimitive group H_{12} of degree 49 and order $(7!)^2$ is a subgroup of one and only one primitive group G, and G is of degree 49 and order $2(7!)^2$. A substitution of G that transforms H_{12} into itself is

$a_2b_1 \cdot a_3c_1 \cdot a_4d_1 \cdot a_5e_1 \cdot a_6f_1 \cdot a_7g_1 \cdot b_3c_2 \cdot b_4d_2 \cdot b_5e_2 \cdot b_6f_2 \cdot b_7g_2 \cdot c_4d_3 \cdot c_5e_3 \cdot c_6f_3 \cdot c_7g_3 \cdot d_5e_4 \cdot d_6f_4 \cdot d_7g_4 \cdot e_6f_5 \cdot e_7g_5 \cdot f_7g_6$.

37. At this point we remove the restriction that the positive subgroup of G is of class > 15. But G contains no substitution of degree 15 and order 5, for the primitive groups of class 15 containing such a substitution are of degree 16 or 17. The substitutions of degree 15 of G are all of order 3 and generate an invariant, and therefore transitive, subgroup H of G of degree > 20. For the determination of H we have precisely the conditions of our former study of the primitive groups of class 15.* The transitive groups H of class 15 (with no substitution of degree 15 and order 5), generated by substitutions of degree 15 and order 3 were found to be:

(1) A simply transitive primitive group of degree 21 simply isomorphic to the alternating group of degree 7.

(2) An imprimitive group of degree 25 simply isomorphic to the direct product of two alternating groups of degree 5.

But there was a serious oversight in that study of the groups of class 15 (as will be shown in § 39) because of which another such group is to be taken into account:

(3) The positive subgroup of the larger transitive constituent of that subgroup of Mathieu's quintuply transitive group of degree 24 in which the subgroup that fixes three letters is invariant.

38. It will now be shown that no G of class 14 contains either (1) or (2).

Suppose the group H of degree 21 is a subgroup of G. Now G contains a substitution of degree 14 and order 2 which transforms H into itself and therefore with H generates a primitive group of degree 21 and order $7!$. It is easy to show that this G is of class < 14. Let G_1 and H_1 be the subgroups of G and H, respectively, which fix one letter. The subgroup H_1 is known. It is isomorphic to the symmetric group of degree 5, a complete group. Then G_1 is a direct product of H_1 and a subgroup of G_1 of order 2. Of the two transitive constituents of degree 10 of H_1, one is primitive and the other is imprimitive, so that G_1 is also intransitive, with one constituent, at least, primitive. This primitive constituent of G_1 can have no invariant substitution except the identity, and therefore remains of order 120 in G_1, as in H_1. The invariant substitution of G_1 is therefore of degree 10 on the letters of the other transitive constituent.

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* *American Journal of Mathematics*, Vol. 39 (1917), p. 281.

As to (2), the imprimitive group of degree 25 and order 3600, it has been proved * that the only primitive groups of degree 25 in which it is invariant are of class 10 or 15.

39. In order to conclude this investigation of the primitive groups of class 14, it is necessary to correct the error made by the author in line 20, page 284, of the paper on primitive groups of class 15. The statement "nor is H an imprimitive group of degree 16; for then its order is of necessity 48, and this case has been examined" is false. The conclusion referred to is in the second and third lines from the bottom of page 283, "the primitive groups under consideration do not contain a transitive group of degree 16 and order 48", quite true because the groups in question are generated by two substitutions of degree 15 and order 3. But the reasoning is not applicable to the groups of the next page, which have three, instead of two, generators of degree 15 and order 3. This paper was written up completely as the research progressed. It was laid aside for some weeks at a point between these two statements and on resuming the task the last quoted result was misinterpreted.

The lines quoted from page 284 are to be replaced by §§ 40-48 below.

40. The group D is tetraedral, with one transitive constituent of degree 12 and one of degree 4:

$$D = \{s_1,\ s_2 = a_1b_2c_3\cdot b_1a_2d_3\cdot c_1d_2a_3\cdot d_1c_2b_3\cdot e_1e_3x\}.$$

For the larger constituent, its conjoin might be used, but it is conjugate to the given regular group.

The group $H = \{D, s_3\}$ is transitive. Suppose it to be of degree 16. It is imprimitive. It is not in a primitive group of lower degree than 21, and then only by virtue of having 5 systems of imprimitivity of four letters each with a letter (x, say) in common. Each of the systems to which x belongs is left fixed by s_1, and these 5 systems can only be $xe_1e_2e_3$, $xa_1a_2a_3$, $xb_1b_2b_3$, $xc_1c_2c_3$, $xd_1d_2d_3$. The group $\{s_1, s_3\}$ is of order 12 and like D has transitive constituents of degrees 12 and 4. The letter following x in s_3 is one of the 12 letters of the large constituent of D, and by transformation we can make it d_2. Then $s_3 = (d_1d_3x)(d_2)(e_1 —)(e_2 —)(e_3 —)\cdots$. The 20 systems of imprimitivity of H are completely determined by D. They are

$$\begin{array}{lllll} xa_1a_2a_3, & xb_1b_2b_3, & xc_1c_2c_3, & xd_1d_2d_3, & xe_1e_2e_3, \\ e_1b_2d_3c_1, & e_1a_2c_3d_1, & e_1d_2b_3a_1, & e_1c_2a_3b_1, & d_1a_1b_1c_1, \\ e_3c_3b_1d_2, & e_3d_3a_1c_2, & e_3a_3d_1b_2, & e_3b_3c_1a_2, & d_2a_2b_2c_2, \\ e_2d_1c_2b_3, & e_2c_1d_2a_3, & e_2b_1a_2d_3, & e_2a_1b_2c_3, & d_3a_3b_3c_3. \end{array}$$

* *American Journal of Mathematics*, Vol. 39 (1917), pp. 308, 309.

It may be well to note that after the first 17 systems are written down from D, the remaining system $d_1, \cdots$ has no letter except d_1 in common with any of the preceding systems, and therefore is $d_1a_1b_1c_1$. Likewise we have the systems $d_2a_2b_2c_2$ and $d_3a_3b_3c_3$. Now s_3 replaces the system $xa_1a_2a_3$ by $e_2d_1c_2b_3$, the latter by $e_1b_2d_3c_1$, and similar statements can be made for the systems $xb_1b_2b_3, \cdots, xc_1c_2c_3, \cdots$. Then

$$s_3 = (d_1d_3x)(a_1b_3e_1)(c_1a_3e_2)(b_1c_3e_3)(a_2c_2b_2),$$

completely determined by the 20 systems of imprimitivity of H.

41. This transitive group H has an elementary subgroup E of order 16 generated by

$$\begin{aligned} e_1 &= s_1^2s_2 = a_1c_1\cdot a_2b_2\cdot b_1d_1\cdot c_2d_2\cdot a_3d_3\cdot b_3c_3\cdot e_1x\cdot e_2e_3, \\ d_1 &= s_1^2s_3 = a_1e_2\cdot a_2b_3\cdot b_1e_1\cdot b_2c_3\cdot c_1e_3\cdot c_2a_3\cdot d_1x\cdot d_2d_3, \\ e_3 &= s_2s_1^2 = a_1b_1\cdot a_2d_2\cdot b_2c_2\cdot c_1d_1\cdot a_3c_3\cdot b_3d_3\cdot e_3x\cdot e_1e_2, \\ d_3 &= s_3s_1^2 = a_1b_2\cdot a_2c_1\cdot b_1c_2\cdot b_3e_3\cdot c_3e_2\cdot e_1a_3\cdot d_1d_2\cdot d_3x. \end{aligned}$$

As a notation for the 15 substitutions of the regular group E we use the letter which follows x in that substitution. Now s_1 transforms $\{e_1, e_3\}$ and $\{d_1, d_3\}$ each into itself. Moreover, $\{E, s_1\} = \{s_1, s_2, s_3\} = H$. Then E is invariant in H and H is of order 48. H has 16 conjugate subgroups of order 3.

That subgroup of the group of isomorphisms of the elementary group E, in which s_1 is invariant, is the transitive group

$$\begin{aligned} I_s = \{s_1, & a_1b_1\cdot a_2b_2\cdot a_3b_3\cdot c_1d_1\cdot c_2d_2\cdot c_3d_3, \\ & a_1c_1\cdot a_2c_2\cdot a_3c_3\cdot b_1d_1\cdot b_2d_2\cdot b_3d_3, \\ & a_1e_1\cdot a_2e_2\cdot a_3e_3\cdot b_1d_3\cdot b_2d_1\cdot b_3d_2, \\ & a_1b_1\cdot a_2b_3\cdot a_3b_2\cdot c_2c_3\cdot d_2d_3\cdot e_1e_2\}, \end{aligned}$$

of order 360. These generators transform $\{E, s_1\}$ into itself. The group of isomorphisms of the elementary group of order 16 is isomorphic to the alternating-8 group, in which $\{(abc)\}$ is invariant in a subgroup of order 360, and $\{(abc)(def)\}$ is invariant in a subgroup of order 36, so that the order of I_s is right.

42. In the imprimitive group of degree 20 towards which we are working there are 5 systems of four letters each permuted according to the alternating-5 group. We can take for the substitution which with H generates this imprimitive group (call it H_1) a substitution of E and transform it into a substitution t which fixes an arbitrary one of the 5 systems and transforms the new system $y_1y_2y_3y_4$ into the system $x \cdots$. Transformation of H and t by substitutions of I_s permits us to write

$$t = (y_4x)(y_1e_1)(y_2e_2)(y_3e_3)(a_1—)(b_1—)(c_1—)(d_1—)(a_3)(b_3)(c_3)(d_3).$$

The four blanks in t are to be filled by a_2, b_2, c_2 and d_2 in some order as yet unknown. Now $ts_1t=(y_1y_2y_3)(x)(e_1)(e_2)(e_3)(y_4)\cdots$ is commutative with s_1, and therefore one of its cycles is the inverse of one of the cycles of s_1. The same is true if for s_1 we put $s_2=(a_1b_2c_3)(b_1a_2d_3)(d_1c_2b_3)(c_1d_2a_3)(e_1e_3x)$, $(a_1c_2d_3)(b_1d_2c_3)(c_1a_2b_3)(d_1b_2a_3)(e_1xe_2)$, or $(a_1d_2b_3)(b_1c_2a_3)(c_1b_2d_3)(d_1a_2c_3)(e_2xe_3)$, all of D. Then one cycle of t is a_1a_2, b_1b_2, c_1c_2, or d_1d_2. Transformation by the substitution $a_1b_1\cdot a_2b_2\cdot a_3b_3\cdot c_1d_1\cdot c_2d_2\cdot c_3d_3$, $a_1d_1\cdot a_2d_2\cdot a_3d_3\cdot b_1c_1\cdot b_2c_2\cdot b_3c_3$, or $a_1c_1\cdot a_2c_2\cdot a_3c_3\cdot b_1d_1\cdot b_2d_2\cdot b_3d_3$ of I_8 enable us to fix upon b_1b_2 as this cycle. The reason for this choice is that it seems to be the most convenient when we come later to connect our groups with subgroups of the quintuply transitive group of degree 24. From s_2 we see that t has either (d_1c_2) or (c_1d_2); from e_1s_1, either (a_1c_2) or (c_1a_2); from b_3s_1, either (d_1a_2) or (a_1d_2). If $t=(a_1c_2)\cdots$, the other cycles are determined by these considerations. Thus t may be

$$b_1b_2\cdot a_1c_2\cdot c_1d_2\cdot d_1a_2\cdot y_1e_1\cdot y_2e_2\cdot y_3e_3\cdot y_4x.$$

But if this t is multiplied by the substitution b_3 of E the product is of degree 12. Then

$$t=a_1d_2\cdot b_1b_2\cdot c_1a_2\cdot d_1c_2\cdot e_1y_1\cdot e_2y_2\cdot e_3y_3\cdot xy_4,$$

uniquely.

43. Consider the imprimitive group $\{E, t\}$. It has the same 5 systems of four letters each as H_1, and permuted according to the alternating group. There is a subgroup $\Delta=\{e_1, e_3, te_1t, te_3t\}$ in $\{E, t\}$, an elementary group of order 16 with 5 regular axial constituents, obviously invariant in $\{E, t\}$. Another subgroup of $\{E, t\}$ is $\Gamma=\{a_2, a_3, t\}$, of which one transitive constituent is the simple group $\{(b_3e_3)(d_1c_2), (b_3d_1)(c_2e_2), (d_1c_2)(e_2y_2)\}$ of order 60. The other transitive constituent is of degree 15 and order 60, and because of the systems of imprimitivity of $\{E, t\}$, Γ is not a direct product. Since Δ and Γ are permutable groups and have only the identity in common, $\{\Delta, \Gamma\}$ is of order 960. Now $\{\Delta, \Gamma\}=\{e_1, e_3, te_1t, te_3t, a_2, a_3, t\}$ $=\{E, t\}$.

Because $s_1^2Es_1=E$ and $s_1^2ts_1=c_1ta_2tc_1$, $\{E, t\}$ is an invariant subgroup of $H_1=\{E, s_1, t\}$, and the latter is therefore of order 2880. The head of H_1 is the intransitive subgroup $\Delta_8=\{\Delta, V\}$ of order 48, where $V=ts_1ts_1=a_1d_1c_1\cdot a_2d_2c_2\cdot a_3d_3c_3\cdot e_1e_2e_3\cdot y_1y_2y_3$.

The order of the subgroup F of H_1 that fixes y_4 is 144. F has two transitive constituents, one on $y_1y_2y_3$ and the other on the remaining 16 letters.

44. If H_1 is the subgroup that fixes one letter of the doubly transitive group H_2, there exists a substitution $T=(y_iz)\cdots$ which transforms H into H. T can be assumed to be a transform of one of the substitutions of E. If T displaces x, its transform by one of the substitutions of E will fix x. Since $THT=H$, T replaces no letter of H by a y_1, y_2, y_3, or y_4. Then $T=(y_iz)(x)\cdots$ transforms s_1 into s_1 or s_1^2, and 6 of its 8 cycles come from I_8; the only cycle unaccounted for must be (yy), with subscripts to be determined. The substitutions of order 2 of I_8 which permute four cycles of s_1 fall into two sets of conjugates under the substitutions of Δ_8 that fix x (the head of H_1 was called Δ_8), with the three that fix the cycle $(e_1e_2e_3)$ of s_1 in one set, and the 12 that displace $(e_1e_2e_3)$ in the other set. For example, one substitution of Δ_8 is $te_3s_1ts_1=V_2=b_1c_1d_1\cdot b_2c_2d_2\cdot b_3c_3d_3\cdot e_1e_2e_3\cdot y_1y_3y_4$. Another is $V_1=a_1b_1d_1\cdot a_2b_2d_2\cdot a_3b_3d_3\cdot e_1e_2e_3\cdot y_2y_4y_3$, the transform of s_1ts_1t by te_1t. Since all the substitutions of Δ_8 that fix x are commutative with s_1, and transform E into E, we need try for T only two of these 15 possible substitutions of I_8. Similarly, of the 10 substitutions of I_8 which permute only two cycles of s_1, the 6 which fix $(e_1e_2e_3)$ are conjugate under the substitutions of Δ_8 which fix x, and the four which displace $(e_1e_2e_3)$ are conjugate under the same subgroup of the head. There are then four possibilities for T:—

$$
\begin{aligned}
T_0&=a_1e_1\cdot a_2e_2\cdot a_3e_3\cdot b_1d_3\cdot b_2d_1\cdot b_3d_2\cdot yy\cdot yz,\\
T_1&=a_1b_1\cdot a_2b_2\cdot a_3b_3\cdot c_1d_1\cdot c_2d_2\cdot c_3d_3\cdot yy\cdot yz,\\
T_2&=a_1e_3\cdot a_2e_2\cdot a_3e_1\cdot b_2b_3\cdot c_1c_3\cdot d_1d_2\cdot yy\cdot yz,\\
T_3&=a_1b_1\cdot a_2b_3\cdot a_3b_2\cdot c_2c_3\cdot d_2d_3\cdot e_1e_2\cdot yy\cdot yz.
\end{aligned}
$$

If $te_2t=a_1b_1\cdot c_1d_1\cdot a_2b_2\cdot c_2d_2\cdot a_3b_3\cdot c_3d_3\cdot y_1y_3\cdot y_2y_4$ of Δ is multiplied by T_1, the product is of degree $\leqq 5$ and is not the identity. If te_2t is multiplied by T_3, the product is $(c_1d_1)\cdots(z\cdots)\cdots$, with a square not the identity and yet of degree $\leqq 5$. Multiply T_2 by the substitution V_1 of Δ_8; the product is

$$(a_1e_1b_3d_2)(a_2e_3b_1d_1)(a_3e_2b_2d_3)(c_2c_3)(zy)(yy)(y_2y_4y_3).$$

The product of $(zy)(yy)$ by $(y_2y_4y_3)$, where the undetermined subscripts are 1, 2, 3, or 4, is a substitution of the alternating-5 group and therefore not a cycle of four letters. It is of order 2, 3, or 5, and the degree of the second or fourth power of the partial product is $\leqq 12$, and is not the identity.

In T_0 the subscripts of the y's are to be determined. Consider $V_1T_0=(a_1d_3e_3)(a_2d_1e_1)(a_3d_2e_2)(b_1b_2b_3)(y_2y_4y_3)(yy)(yz)$. The partial product of $(y_2y_4y_3)$ by $(yy)(yz)$ must be of order 3 and therefore T_0 fixes y_1. $V_2T_0=(a_1e_1a_2e_2a_3e_3)(b_1c_1b_2c_2b_3c_3)(d_1d_2d_3)\cdots$. The product of $(y_1y_3y_4)$ by $(yy)(yz)(y_1)$ must be of order 2 and degree 4, hence is $(y_1y_4)(z\text{-})$, so that

$$T_0 = y_3y_4 \cdot y_2z \cdot a_1e_1 \cdot a_2e_2 \cdot a_3e_3 \cdot b_1d_3 \cdot b_2d_1 \cdot b_3d_2.$$

After transformation by te_2t, we have finally the unique substitution

$$T = y_4z \cdot y_1y_2 \cdot b_1e_1 \cdot b_2e_2 \cdot b_3e_3 \cdot a_1c_3 \cdot a_2c_1 \cdot a_3c_2.$$

45. It must next be proved that H_1 is indeed the subgroup of $H_2 = \{H_1, T\}$ that fixes z. Now

$$H_1 = F_8 + F_8tF_8 + F_8te_2tF_8,$$

where F_8 is the subgroup of H_1 that fixes y_4. We have to show that $TF_8T = F_8$, and that $(tT)^3$ and $(te_2tT)^3$ are substitutions of F_8. Now $F_8 = \{H, V = s_1ts_1t\}$. But $THT = H$, as we know, and $TVT = s_1V^2$. Also,

$$tT = xzy_4 \cdot b_1e_2y_1 \cdot b_2e_1y_2 \cdot b_3e_3y_3 \cdot a_1d_2c_3 \cdot a_3c_2d_1$$

and

$$te_2tT = y_4y_3z \cdot a_1a_2a_3 \cdot b_1d_1e_1 \cdot b_2d_2e_2 \cdot b_3d_3e_3 \cdot c_1c_3c_2,$$

so that $(tT)^3 = 1$ and $(te_2tT)^3 = 1$. Then H_2 exists. Since there is no positive doubly transitive group of order $21 \cdot 20 \cdot 144$ of class < 15, H_2 is certainly of class 15, an essential point not previously proved.

Since $s_1T = Ts_1$, the doubly transitive group $\{E, t, T\}$ is invariant in H_2. Now $F = \{E, Y\}$ where

$$Y = tc_1ta_2te_2 = a_1a_2e_2 \cdot b_1b_2x \cdot c_1c_2e_1 \cdot d_1d_2e_3 \cdot a_3d_3c_3 \cdot y_1y_3y_2,$$

is the subgroup of $\{E, t\}$ that fixes y_4, and

$$\{E, t\} = F + FtF + Fte_2tF;$$

while $TET = E$, and $TYT = c_1Y^2$. We have as above $(tT)^3 = (te_2tT)^3 = 1$, so that $\{E, t, T\}$ is of order $21 \cdot 20 \cdot 48$.

46. In this connection it is easy to prove the existence of Mathieu's quintuply transitive group of degree 24.* We shall show that the group $\{E, t, T\}$ whose existence has just been proved is the subgroup that leaves three letters of Mathieu's group fixed, and that H_2 is a subgroup of the transitive constituent of the largest subgroup of Mathieu's G^{24} in which $\{E, t, T\}$ (or say G^{21}) is invariant.

The generating substitutions of G^{24} given by Mathieu are in § 18 above. Since the connection with the immediately preceding work must be made clear, we now replace Mathieu's letters x_i, or rather the subscripts i ($i = \infty$,

* Mathieu, *Liouville's Journal*, Vol. 18 (1873), p. 25.

Miller, *Messenger of Mathematics*, Vol. 27 (1897), p. 187; *Bulletin de la Société Mathématique de France*, Vol. 28 (1900), p. 206.

De Séguier, *Groupes de substitutions* (1912), pp. 156-163.

$0, 1, \cdots, 22)$, by $\infty, 0, w, y_4, y_3, a_1, z, b_3, b_1, c_1, e_1, a_2, e_2, d_2, e_3, y_2, d_1, c_2, y_1, a_3, b_2, c_3, x, d_3$, respectively. Then Mathieu's $\{Z_1, Z_2, Z_3, Z_4\}$ is precisely our group E, for $Z_1 = d_3$, $Z_2 = b_3$, $Z_3 = a_2$, $Z_4 = c_3$, and his Y is our Y. Moreover,

$$\begin{aligned} U &= y_4c_2e_1b_3c_1 \cdot y_3d_2e_3a_3a_1 \cdot y_1a_2e_2d_3b_1 \cdot y_2b_2xc_3d_1 \\ &= (b_2Yb_2te_1)^3, \end{aligned}$$

a substitution of $\{E, t\}$. And t is in $\{E, Y, U\}$, for $t = b_2Y^2b_2U^2e_1$. Thus it is proved that $\{E, t\} = \{E, Y, U\}$.

Mathieu gives no substitution that with G^{20} generates G^{21}, but the substitution $YBa_3B^{-1}Y^{-1}$ fixes ∞, 0 and w, and is our T. From Mathieu's set of generators we also get

$$\begin{aligned} T' &= Y^{-1}B^{-7}c_2B^7Y \\ &= zw \cdot y_1y_2 \cdot a_2d_3 \cdot a_3b_2 \cdot b_1d_1 \cdot b_3c_2 \cdot c_3d_2 \cdot e_2e_3, \\ T'' &= A^{-2}d_3A^2b_3 \\ &= wo \cdot y_1y_2 \cdot a_1c_2 \cdot a_2c_1 \cdot a_3c_3 \cdot b_1b_2 \cdot d_1d_2 \cdot e_1e_2, \\ T''' &= Ye_2TA^{-1}XT''XATe_2tY^{-1}e_2 \\ &= 0\,\infty \cdot y_1y_2 \cdot a_1c_1 \cdot a_2c_3 \cdot a_3c_2 \cdot b_2b_3 \cdot d_2d_3 \cdot e_2e_3. \end{aligned}$$

Since

$$\begin{aligned} T'''T'' &= wo\,\infty \cdot a_1a_2a_3 \cdot b_1b_2b_3 \cdot c_1c_2c_3 \,.\, d_1d_2d_3 \cdot e_1e_2e_3, \\ T''T' &= zwo \cdot a_1b_3c_2 \cdot a_2c_1d_3 \cdot a_3d_2b_1 \cdot b_2d_1c_3 \cdot e_1e_3e_2, \\ T'T &= y_4zw \cdot a_1c_3d_2 \cdot a_2d_3c_1 \cdot a_3e_2b_3 \cdot b_1d_1e_1 \cdot b_2c_2e_3, \end{aligned}$$

the existence of G^{24} will be proved as soon as it is shown that

$$\begin{aligned} &T'\{E, t\}T' = \{E, t\}, \\ &T''\{E, t, T\}T'' = \{E, t, T\}, \\ &T'''\{E, t, T, T'\}T''' = \{E, t, T, T'\}. \end{aligned}$$

This is established as follows:

$$\begin{aligned} &T'ET' = T''ET'' = T'''ET''' = E, \\ &T'tT' = T'''tT''' = tb_3Y^2b_3,\ T''tT'' = t, \\ &T''TT'' = T'''TT''' = T, \\ &T'''T'T''' = T'. \end{aligned}$$

In the course of the above proof of the existence of the quintuply transitive group G^{24} we found a substitution $T'''T''$ of order 3 which transforms $\{E, t, T\}$ into itself, and which except for the cycle $(wo\,\infty)$ is our s_1. The transitive constituent of degree 21 of $\{G^{21}, T'''T''\}$ is H_2.

It should be noted that G^{24} is of class 16.

47. It remains to be seen if H_2 is a subgroup of a larger primitive group of class 15. Suppose it a subgroup of a group G of the same degree. Because of H, G is not triply transitive, and G_1 (the subgroup that fixes z) has the same 5 systems of imprimitivity as H_1. The transitive constituents

of the head of G_1 cannot be symmetric groups, for then the class of G would be $\leqq 10$. If the 5 alternating-4 groups of the head are combined to make its order > 48, there must be more substitutions of order 2 and of degree 16 or 20, and this is impossible because such a substitution would have at least four cycles in common with a substitution of Δ and the resulting product would be of degree $\leqq 12$. The head of G_1 is Δ_8 of order 48. Then the group in the systems is the symmetric group of order 120.

In H_1 there are 96 subgroups of order 5, and in H_1, $\{U\}$, for example, is invariant in the subgroup $\{U,\ c_3,\ e_2V_1e_2\}$ of order 30. According to Sylow's Theorem, G_1 has 96 subgroups of order 5 and each is invariant in a subgroup I_u of order 60. Suppose that I_u has substitutions which transform U into U^2 without permutation of cycles. One such substitution (Q) fixes y_4 and therefore as a substitution of the imprimitive group G_1 permutes y_1, y_2 and y_3 among themselves, that is, fixes y_1, y_2 and y_3. Therefore

$$Q = c_2e_1c_1b_3 \cdot d_2e_3a_1a_3 \cdot a_2e_2b_1d_3 \cdot b_2xd_1c_3.$$

Now $b_2Q = a_1a_2c_1d_2b_1c_2 \cdot a_3b_3d_3 \cdot e_1e_2e_3 \cdot b_2d_1$, and $(b_2Q)^3$ is of degree 8. Then all the substitutions of I_u not in the invariant subgroup $\{U,\ c_3\}$ permute cycles of U according to a group of order 6, the symmetric-3 group. A substitution of I_u which permutes two cycles of U cannot be commutative with U, for then it or its fifth power would be of degree 10. The known subgroup $\{U,\ e_2V_1e_2\}$ of I_u shows that the cycle $(y_1 \cdot \cdot \cdot)$ of U is fixed by I_u. Then a substitution which permutes two cycles of U is $W = (y_2y_3)(y_4)(y_1) \cdot \cdot \cdot$, with the property of transforming U into U^4 or U^2. No one of the 30 substitutions of I_u which permute two cycles of U is commutative with U. Now c_3 transforms U into its inverse. If W does the same, Wc_3 is commutative with U, and therefore W is in $\{U,\ c_3,\ e_2V_1e_2\}$. Then $W^{-1}UW = U^2$. The subgroup $\{U,\ W\}$ has one transitive constituent of degree 10 and two of degree 5 each. The transitive constituent of degree 10 is completely determined by U and the given cycle (y_2y_3) of W, and W is completely determined as to the other two constituents because y_1 and y_4 are fixed by it. Thus

$$W = c_2e_1c_1b_3 \cdot a_2e_2b_1d_3 \cdot y_2y_3 \cdot d_2xa_1c_3 \cdot b_2e_3d_1a_3.$$

Since $W^2 \doteq c_3$, $W^3EW = E$, $W^3s_1W = a_2s_1$, $W^3tW = a_1ta_1$, $W^3TW = VTV^2$, $\{H_2,\ W\}$ is a doubly transitive group of order $21 \cdot 20 \cdot 288$. There is no doubly transitive group of this order among the groups of class < 14. Now

$$Vd_2WV^2 = a_1c_2 \cdot a_2c_1 \cdot a_3c_3 \cdot b_1b_2 \cdot c_1c_2 \cdot e_1e_2 \cdot y_1y_2,$$

so that $\{H_2,\ W\}$ is actually of class 14. There is therefore no group of class 15 and degree 21 of which our H_2 is a subgroup.

The substitution Vd_2WV^2 above is $(wo)T''$, so that $\{H_2, W\}$ is the transitive constituent of degree 21 of the largest subgroup $\{E, t, T, T'', T'''\}$ of G^{24} in which $\{E, t, T\}$ is invariant. We have shown that $\{H_2, W\}$ is the only primitive group of class 14 in which H_2 is invariant.

48. Can H_2 be the subgroup that leaves one letter fixed of a triply transitive group H_3 of degree 22? If H_3 exists, there is a substitution $T_1 = (zw)(y_4) \cdots$ of order 2 and degree 16 in H_3 which transforms $H_1 = \{E, S_1, t\}$ into itself. Now H_1 has only one system of imprimitivity of four letters including y_4, $y_1y_2y_3y_4$, and therefore T_1 permutes the three letters y_1, y_2 and y_3 among themselves. T_1 can be chosen to fix x. This T_1 transforms E into E and $\{s_1\}$ into $\{s_1\}$. Then T_1 has 6 cycles from I_8 and may now be written $(zw)(y_1y_2)(y_4)(y_3)(x) \cdots$. Since T_1 respects the systems of four letters each of H_1, only three substitutions of order 2 of I_8 are available for T_1: $a_1b_1 \cdot c_1d_1 \cdot a_2b_2 \cdot c_2d_2 \cdot a_3b_3 \cdot c_3d_3$, $a_1c_1 \cdot b_1d_1 \cdot a_2c_2 \cdot b_2d_2 \cdot a_3c_3 \cdot b_3d_3$ and $a_1d_1 \cdot b_1c_1 \cdot a_2d_2 \cdot b_2c_2 \cdot a_3d_3 \cdot b_3c_3$. But these three substitutions occur in substitutions of Δ, which, multiplied by T_1, would lower the class of H_3 to 4 or 6. There is no group H_3 of degree 22 with H_2 as a subgroup of index 22.

49. At line 32 of page 288 of the paper on Class 15 all the partitions of the degree of the subgroup F have been rejected except two: 12, 9 and 9, 12. There is a slip in the discussion of these two cases which may be corrected by noting that the conditions of the following theorem are fulfilled. By the notation $G(x)$ is meant the subgroup of G that leaves fixed the letter x, and by $G(x)(a)$ the subgroup of $G(x)$ that fixes a.

Let G be a transitive group whose $G(x)$ has just two transitive constituents, A on the $m + h$ letters $a, a', \ldots$ and B of a lower degree m. Let $G(x)(a)$, of degree $\geqq 2m$, have no constituent of degree $< h$, but at least one transitive constituent of degree m. Then G is imprimitive.

The theorem is true if $m < 3$.* Then it is assumed that $m \geqq 3$. Let $b_1, b_2, \cdots, b_m$ be the letters of B.

Suppose the theorem false. Then $G(x)$ is a maximal subgroup of G. Since A is the only constituent of $G(x)$ of its degree, it is paired with itself and G contains a substitution $S = (xa) \cdots$† which transposes $G(x)$ and $G(a)$ and transforms $G(x)(a)$ into itself. S cannot permute the m letters

* Miller, *Proceedings of the London Mathematical Society*, Vol. 38 (1897), p. 533.

† Burnside, *Proceedings of the London Mathematical Society*, Vol. 33 (1901), p. 162.

of B among themselves for then $G(a)$ and $G(x)$ would generate an intransitive group, a subgroup of G, contrary to the assumption that $G(x)$ is maximal. If S transforms the given transitive constituent of degree m of $G(x)(a)$ into itself, its letters are certainly $a_1, a_2, \cdots, a_m$ of the constituent A. All the other $h-1$ letters $a', \cdots$ of A are fixed by $G(x)(a)$, for if displaced they would occur in transitive constituents of degree $< h$, contrary to hypothesis. Then because the degree of $G(x)(a)$ is $\geqq 2m$, it displaces all the letters of B and permutes them among themselves, which is impossible.

But $G(x)(a)$ may have two transitive constituents of degree m, transposed by S. Then one is on $b_1, b_2, \cdots, b_m$ and the other is on $a_1, a_2, \cdots, a_m$. Now $h \geqq 2$, for if $h=1$, A is doubly transitive, which is impossible when G is primitive.* The other letters of $G(x)$ are as before all fixed by $G(x)(a)$, so that $G(x)(a) = G(x)(a')$. In this case both $G(a)$ and $G(a')$ have transitive constituents of degree m on $a_1, a_2, \cdots, a_m$. The group generated by $G(a)$ and $G(a')$ is intransitive. But $G(a)$, a conjugate of $G(x)$, is a maximal subgroup of G.

Hence $G(x)$ cannot be a maximal subgroup of G, and G is therefore imprimitive.

STANFORD UNIVERSITY, CALIFORNIA.

* *Transactions of the American Mathematical Society*, Vol. 20 (1919), p. 66, Th. 16.

Summability of Infinite Products.

By George M. Robison.

1. *Introduction.* Among the definitions giving a value to a divergent series are those which are expressed by means of a linear transformation. Let $s_1, s_2, \cdots, s_n, \cdots$ be the sequence corresponding to the infinite series $\sum_{n=1}^{\infty} u_n$. Form a new sequence t_n, where $t_n = \sum_{k=1}^{n} a_{n,k} s_k$. If the limit of t_n, as n becomes infinite, exists, we say that this limit is the generalized value of the series $\sum_{n=1}^{\infty} u_n$ by the transformation $A:(a_{n,k})$. For such a definition to be useful, it must include convergence; that is, if we apply it to a convergent series, the value assigned to the series must be the same as the convergent value. If a transformation A satisfies this condition, we say that it is regular. The conditions, which the terms $(a_{n,k})$ of a transformation must satisfy in order that it be regular, are stated in the Silverman-Toeplitz Theorem.*

The theory of infinite products furnishes us an analogous problem. From the given infinite product $\prod_{n=1}^{\infty} (1 + u_n)$, form a sequence P'_n as follows: $P'_n = \prod_{k=1}^{n} (1 + a_{n,k} \cdot u_k)$, where $a_{n,k}$ is the element in the nth row and kth column of a triangular matrix of numbers defining the transformation A. If $\lim_{n\to\infty} P'_n$ exists, we define this limit to be the value assigned to the given infinite product by the transformation $A:(a_{n,k})$. Let us consider the following illustration:

To the infinite product $\prod_{n=1}^{\infty} [1 + (-2)]$ let us apply the transformation $A:(a_{n,k} = n - k + 1/n)$. The resulting sequence is as follows:

$$P'_n = 0, \qquad n \text{ even},$$
$$P'_n = (-1)^{(n+1)/2} 1^2 \cdot 3^2 \cdot 5^2 \cdots (n-1)^2/n^{n-1} \qquad n \text{ odd},$$

where $\lim_{n\to\infty} P'_n = 0$. Thus, zero is the value assigned to the above divergent infinite product by the transformation $A:(a_{n,k} = n - k + 1/n)$.

2. *Regularity of transformations.* A transformation $A:(a_{n,k})$ is said to

* L. L. Silvermann, Missouri Dissertation, (1910); Toeplitz, *Prace Matematyczno-fizyczne*, Vol. 22 (1911), p. 113.

be regular, if it gives to every convergent infinite product the value to which it converges. With regard to the regularity of these transformations, we have the following theorems:

THEOREM I. *A necessary condition that a transformation $A:(a_{n,k})$ be regular is*

$$1)\quad \lim_{n\to\infty} a_{n,k} = 1 \qquad \textit{for each } k;$$

$$2)\quad |a_{n,k}| < M,$$

where M is some constant.

Proof. Let us consider the infinite product $\prod_{n=1}^{\infty}(1+u_n)$, where $u_n = u_k$ for $n = k$; $u_n = 0$, $n \neq k$. Thus, $P_n = \prod_{k=1}^{n}(1+u_n) = 1 + u_k$, if $n > k$, otherwise $P_n = 1$. Applying transformation $A:(a_{n,k})$, we have

$$P'_n = 1 + a_{n,k}\cdot u_k, \qquad n > k.$$

From definition, $P = 1 + u_k$. If $\lim_{n\to\infty} P'_n = P$, it is necessary that $\lim_{n\to\infty} a_{n,k} = 1$.

To show that condition (2) is necessary:

Under the assumption that the transformation $A:(a_{n,k})$ is not bounded we shall transform a convergent product into a divergent sequence. We shall assume that the elements of the transformation satisfy condition (1) of the theorem. Hence, we can find a set of integers

$$r_0 < n_0 < r_1 < n_1 < r_2 < n_2 < \cdots$$

such that

$$\begin{aligned} &|a_{n_0,r_0}| > 4, \\ &|a_{n_1,r_1}| > 4^3, \qquad |a_{n_1,r_0}| < 2, \\ &|a_{n_2,r_2}| > 4^5, \qquad |a_{n_2,r_1}| < 2, \qquad |a_{n_2,r_0}| < 2, \end{aligned}$$

and in general,

$$|a_{n_i,r_i}| > 4^{2i+1}, \quad |a_{n_i,r_l}| < 2, \qquad l = 0, 1, 2, \cdots, i-1.$$

We shall transform, by the above transformation, the convergent infinite product $\prod_{n=1}^{\infty}(1+u_n)$, where

$$\begin{aligned} &u_n = 0, \quad n \neq r_0, \quad n \leq n_0; \quad u_{r_0} = (1/4)\operatorname{sgn} a_{n_0,r_0} \\ &u_n = 0, \quad n \neq r, \quad n_0 < n \leq n_1; \quad u_{r_1} = (1/8)\operatorname{sgn} a_{n_1 r_1}. \end{aligned}$$

In general

$$u_n = 0, \quad n \neq r_i, \quad n_{i-1} < n \leqq n_i; \quad u_{r_i} = (\tfrac{1}{2}^{i+2})\operatorname{sgn} a_{n_i,r_i}.$$

We shall show that a certain sub-sequence of the sequence P'_n has no limit.

$$P'_{n_0} = \prod_{r=1}^{n_0} (1 + a_{n_0,r} \cdot u_r) = 1 + a_{n_0,r_0} u_{r_0} = 1 + (1/4) \cdot | a_{n_0,r_0} | > 2,$$

$$P'_{n_1} = \prod_{r=1}^{n_1} (1 + a_{n_1 r} \cdot u_r) = (1 + a_{n_1} u_{r_0})(1 + a_{n_1,r_1} \cdot u_{r_1}),$$

$$| P'_{n_1} | \geq (1/2) \cdot (1 + 8) > 4,$$

$$P'_{n_2} = \prod_{r=1}^{n_2} (1 + a_{n_2,r} \cdot u_r)(1 + a_{n_2,r_0} \cdot u_{r_0})(1 + a_{n_2,r_1} \cdot u_{r_1}) \cdot (1 + a_{n_2,r_2} \cdot u_{r_2}),$$

$$| P'_{n_2} | \geq (1/2) \cdot (3/4)[1 + (4^5/2^4)],$$

and in general

$$P'_{n_i} = (1 + a_{n_i,r_0} \cdot u_{r_0})(1 + a_{n_i,r_1} \cdot u_{r_1})(1 + a_{n_i,r_2} \cdot u_{r_2}) \cdots (1 + a_{n_i,r_i} \cdot u_{r_i}),$$

$$| P'_{n_i} | \geq \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{7}{8} \cdot \cdots \frac{2^i - 1}{2^i}\left(1 + \frac{4^{2i+1}}{2^{i+2}}\right) \geq \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{7}{8} \cdot \cdots \frac{2^{i-1}}{2^i} \cdot \frac{4^{2i+1}}{2^{i+2}}$$

$$\geq \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{7}{8} \cdots \frac{2^{i-1}}{2^i} 2^{2i+1} \cdot 2^{i-1} \geq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} \cdot 2^{2i+1} \cdot 2^{i-1} \geq 2^{2i},$$

$$| P'_{n_i} | \geq 2^{2i}.$$

$\therefore \lim_{i \to \infty} | P'_{n_i} | = \infty$, thus showing the necessity of condition (2).

THEOREM II. *A sufficient condition that* $\lim_{n \to \infty} P_n'$ *shall exist and be equal to* $\lim_{n \to \infty} P_n$, *where* $P_n = \prod_{k=1}^{n} (1 + u_k)$, *and where* $\sum_{k=1}^{\infty} | u_k |$ *converges is that* 1) $\lim_{n \to \infty} a_{n,k} = 1$, *for each* k; 2) $| a_{n,k} | < M$, M some constant.

Proof. Let us write

$$P_n' = \prod_{k=1}^{n} (1 + a_{n,k} \cdot u_k) = \prod_{k=1}^{n} (1 + u_k + \delta_{n,k} \cdot u_k), \qquad \delta_{n,k} = a_{n,k} - 1,$$

$$P_n' = P_n + P_n \cdot \sum_{k=1}^{n} \cdot \delta_{n,k} \cdot [u_k/(1 + u_k)]$$

$$+ P_n \cdot \sum_{\substack{k=1, l=1 \\ k \neq l}}^{m,n} \delta_{n,k} \cdot \delta_{n,l} [u_k \cdot u_l/(1 + u_k)(1 + u_l)]$$

$$+ \cdots + \prod_{k=1}^{n} \delta_{n,k} \cdot u_k.$$

Let us call

$$\sum_{k=1}^{n} \delta_{n,k} \cdot u_k/(1 + u_k) = Q_{n,1}, \text{ etc., and}$$

$$V_n = \sum_{k=1}^{n} | \delta_{n,k} \cdot u_k/(1 + u_k) | = \sum_{k=1}^{n} | \delta_{n,k} \cdot u_k | \cdot 1/| 1 + u_k |$$

$$| Q_{n,1} | \leq V_n, \qquad | Q_{n,2} | \leq V_n^2/2, \cdots, \qquad | Q_{n,p} | \leq V_n^p/p!, \cdots.$$

Substituting in the above, we have

$$|P_n' - P_n| \leq |P_n| \{V_n + (V_n^2/2) + \cdots + (V_n^n/n!)\}$$
$$|P_n' - P_n| \leq |P_n| \cdot [e^{V_n} - 1].$$

A sufficient condition that $\lim_{n\to\infty} P_n' = \lim_{n\to\infty} P_n$, provided the latter limit exists, is that $\lim_{n\to\infty} V_n = 0$. If $\lim_{n\to\infty} P_n$ exists $\neq 0$, $\lim_{n\to\infty} u_n = 0$.

Hence, at most, only a finite number of u_n can be found such that $R(u_n) \leq -1$.

(a) We shall consider only infinite products, where $R(u_n) > -1$

$$\therefore \quad |1 + u_k| > 1 + L, \text{ where } R(u_n) > L > -1$$

$$\therefore \quad V_n = \sum_{k=1}^{n} |\delta_{n,k} \cdot u_k| \, 1/|1 + u_k| \leq 1/1 + L \sum_{k=1}^{n} |\delta_{n,k} \cdot u_k|$$

If $\lim_{n\to\infty} \sum_{k=1}^{n} |\delta_{n,k} \cdot u_k| = 0$, $\lim_{n\to\infty} V_n = 0$.

It can be shown that $\lim_{n\to\infty} \sum_{k=1}^{n} |\delta_{n,k} \cdot u_k|$ is zero, provided $|\delta_{n,k}| < M$ and $\lim_{n\to\infty} \delta_{n,k} = 0$ for each k.

This proof can be modified to hold in the case of a finite number of the $R(u_n)$ being < -1.

Hence, the sufficiency of the above condition.

3. *Summability of Double Infinite Products.* The definitions of the first section can be extended to double infinite products as follows:

Let $P_{m,n} = \prod_{k=1,l=1}^{m,n} (1 + u_{k,l})$ represent the sequence corresponding to the infinite product $\prod_{k=1,l=1}^{\infty\infty} (1 + u_{k,l})$. A new sequence $P'_{m,n}$ is defined by the relation

$$P'_{m,n} = \prod_{k=1,l=1}^{m,n} (1 + a_{m,n,k,l} \cdot u_{k,l}),$$

where $A:(a_{m,n,k,l})$ is a given set of constants. If $\lim_{m,n\to\infty} P'_{m,n}$ exists we define this limit to be the value assigned to the infinite product by the transformation $A:(a_{m,n,k,l})$.

A transformation is said to be regular if, when applied to a convergent infinite product, the value assigned is the same as its convergent value. We shall now prove certain theorems with regard to the regularity of these transformations.

THEOREM III. *A necessary condition that a transformation $A:(a_{m,n,k,l})$ be regular is that*

1) $\lim\limits_{m,n\to\infty} a_{m,n,k,l} = 1$ *for each k and l;*

2) $| a_{m,n,k,l} | < M$ *where M is some fixed constant.*

Proof. Let us consider the infinite product

$$\prod_{m=1,n=1}^{\infty\infty} (1 + u_{mn}) \text{ where } u_{p,r} = u_{p,r} \neq 0,$$

p,r being two fixed integers, all other $u_{m,n} = 0$.

Applying the transformation, we have

$$P'_{m,n} = 1 + a_{m,n,p,r} \cdot u_{p,r}.$$

From definition, $$P = \lim_{m,n\to\infty} P_{m,n} = 1 + u_{p,r}.$$

Therefore, if $\lim\limits_{m,n\to\infty} P'_{m,n} = P$, it is necessary that

$$\lim_{m,n\to\infty} a_{m,n,p,r} = 1.$$

To show that condition (2) is necessary: Under the assumption that the terms of transformation $A:(a_{m,n,k,l})$ are not bounded, we shall transform a convergent product into a divergent sequence. We shall assume that the transformation satisfies condition (1) of the theorem.

Since the condition

$$| a_{m,n,k,l} | < M$$

is not satisfied, we can find a set of integers

$$(m_0, n_0, p_0, r_0), (m_1, n_1, p_1, r_1) \cdots (m_r, n_r, p_r, r_r) \text{ where}$$
$$p_0 < m_0 < p_1 < m_1 < p_2 < m_2 \cdots$$
$$r_0 < n_0 < r_1 < n_1 < r_2 < n_2 \cdots$$

such that

$$
\begin{array}{lll}
| a_{m_0,n_0,p_0,r_0} | > 4, & & \\
| a_{m_1,n_1,p_1,r_1} | > 4^3, & | a_{m_1,n_1,p_0,r_0} | < 2, & \\
| a_{m_2,n_2,p_2,r_2} | > 4^5, & | a_{m_2,n_2,p_1,r_1} | < 2, & | a_{m_2,n_2,p_0,r_0} | < 2,
\end{array}
$$

and in general,

$$| a_{m_i,n_i,p_i,r_i} | > 4^{2i+1}, \quad | a_{m_i,n_i,p_j,r_j} | < 2, \quad j = 0, 1, 2, \cdots, i-1.$$

We shall define the sequence of $u_{m,n}$ as follows:

$$
\begin{array}{lllll}
u_{m,n}=0, & m\neq p_0, & n\neq r_0, & m\leq m_0, & n\leq n_0,\\
u_{m,n}=0, & m\neq p_0, & n=r_0, & m\leq m_0, & \\
u_{m,n}=0, & m=p_0, & n\neq r_0, & n\leq n_0, & \\
u_{mn}=\tfrac{1}{4}\,\mathrm{sgn}\,a_{m_0,n_0,p_0,r_0}, & & & m=p_0,\quad n=r_0, & \\
u_{m,n}=0, & m\neq p_1, & n\neq r_1, & m_0<m\leq m_1, & n_0<n\leq n_1,\\
u_{m,n}=0, & m\neq p_1, & n=r_1, & m_0<m\leq m_1, & \\
u_{m,n}=0, & m=p_1, & n\neq r_1, & n_0<n\leq n_1, & \\
u_{mn}=\tfrac{1}{8}\cdot\mathrm{sgn}\,a_{m_1,n_1,p_1,r_1} & & & m=p_1,\quad n=r_1. &
\end{array}
$$

In general

$$
\begin{array}{lllll}
u_{m,n}=0, & m\neq p_i, & n\neq r_i, & m_{i-1}<m\leq m_i, & n_{i-1}<n\leq n_i,\\
u_{mn}=0, & m=p_i, & n\neq r_i, & n_{i-1}<n\leq n_i, & \\
u_{m,n}=0, & m\neq p_i, & n=r_i, & m_{i-1}<m\leq m_i, & \\
u_{m,n}=\tfrac{1}{2}i+2\cdot\mathrm{sgn}\,a_{m_i,n_i,p_i,r_i}. & & & &
\end{array}
$$

From the above, we have

$$\lim_{m,n\to\infty} P_{mn}=\lim_{m,n\to\infty}\prod_{p=1,r=1}^{m,n}(1+u_{p,r}) \text{ exists, for } \sum_{p=1,r=1}^{\infty\infty}|\,u_{p,r}\,| \text{ converges.}$$

We shall show that a certain sub-sequence of the sequence $P'_{m,n}$ has no limit as m and n become infinite.

$$P'_{m_0,n_0}=\prod_{p=1,r=1}^{m_0,n_0}(1+a_{m_0,n_0,p,r}\cdot u_{p,r})=1+a_{m_0,n_0,p_0,r_0}\cdot u_{p_0,r_0}>2,$$

$$P_{m_1,n_1}=(1+a_{m_1,n_1,p_0,r_0}\cdot u_{p_0,r_0})(1+a_{m_1,n_1,p_1,r_1}\cdot u_{p_1,r_1}),$$

$$|\,P'_{m_1,n_1}\,|\geq\tfrac{1}{2}\cdot(1+8)\geq 4,$$

$$P'_{m_2,n_2}=(1+a_{m_2,n_2,p_0,r_0}\cdot u_{p_0,r_0})(1+a_{m_2,n_2,p_1,r_1}\cdot u_{p_1,r_1})(1+a_{m_2,n_2,p_2,r_2}\cdot u_{p_2,r_2}),$$

$$|\,P'_{m_2,n_2}\,|\geq(1/2)\cdot(3/4)[1+(4^5/2^4)].$$

In general

$$|\,P'_{m_i,n_i}\,|\geq\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{7}{8}\cdots\frac{2^i-1}{2^i}\left(1+\frac{4^{2i+1}}{2^{i+2}}\right)\geq\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{7}{8}\cdots\frac{2^i-1}{2^i}\frac{4^{2i+1}}{2^{i+2}}$$

$$\geq\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{7}{8}\cdots\frac{2^i-1}{2^i}2^{2i+1}\cdot 2^{i-1}\geq\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdots\frac{1}{2}\cdot 2^{2i+1}\cdot 2^{i-1}=2^{2i+1}.$$

$$\therefore\ \lim_{i\to\infty}|\,P'_{m_i,n_i}\,|=\infty.$$

This shows the necessity of condition (4).

THEOREM IV. *A sufficient condition that* $\lim_{m,n\to\infty} P'_{m,n}$ *shall exist and equal to* $\lim_{m,n\to\infty} P_{m,n}$*, where* $P_{m,n} = \prod_{k=1,l=1}^{m,n} (1 + u_{k,l})$ *and where* $\sum_{k=1,l=1}^{\infty\infty} | u_{k,l} |$ *shall converge, and* $R \mid u_{k,l} \mid > -1$ *is that*

1) $\lim_{m,n\to\infty} a_{m,n,k,l} = 1$ *for each* k *anl* l;

2) $\lim_{m,n\to\infty} \sum_{k=1}^{m} | a_{m,n,k,l}{}^{-1} | = 0$ *for each* l;

3) $\lim_{m,n\to\infty} \sum_{l=1}^{n} | a_{m,n,k,l}{}^{-1} | = 0$ *for each* k;

4) $| a_{m,n,k,l} | < K$ *where* K *is some constant.*

Proof. We have

$$P'_{m,n} = \prod_{k=1,l=1}^{n,n} (1 + a_{m,n,k,l} \cdot u_{k,l}) = \prod_{k=1,l=1}^{m,n} (1 + u_{k,l} + \delta_{m,n,k,l} \cdot u_{k,l}).$$

$$a_{m,n,k,l} = 1 + \delta_{m,n,k,l}.$$

$$P'_{mn} = P_{m,n} + P_{m,n} \cdot \sum_{k=1,l=1}^{m,n} \frac{\delta_{m,n,k,l} \cdot u_{k,l}}{1 + u_{k,l}} + P_{m,n} \sum_{k_1=1,l_1=1}^{m,n} \sum_{\substack{k_2=1,l_2=1 \\ (k_1l_1)\neq(k_2l_2)}}^{m,n}$$

(1)
$$\frac{\delta_{m,n,k_1,l_1} \cdot \delta_{m,n,k_2,l_2} u_{k_1,l_1} \cdot u_{k_2,l_2}}{(1 + u_{k_1l_1})(1 + u_{k_2l_2})} + \cdots + \prod_{k=1,l=1}^{n,n} \delta_{m,n,k,l,} \cdot u_{k,l,}.$$

Let us put

$$\sum_{k=1,l=1}^{m,n} \frac{\delta_{m,n,k,l} \cdot u_{k,l}}{1 + u_{k,l}} = Q_{m,n,1}, \text{ etc.}$$

We define

$$V_{m,n} = \sum_{k=1,l=1}^{m,n} \left| \delta_{m,n,k,l} \cdot \frac{u_{k,l}}{1 + u_{k,l}} \right|.$$

Thus

$$| Q_{m,n,1} | \leq V_{m,n};\ | Q_{m,n,2} | \leq \frac{(V_{m,n})^2}{2!} : \cdots | Q_{m,nmn} | \leq \frac{(V_{m,n})^{mn}}{m \cdot n!}$$

Substituting in (1), we have

$$| P'_{m,n} - P_{m,n} | \leq | P_{m,n} | \{ V_{m,n} + \frac{(V_{mn})^2}{2} + \frac{(V_{mn})^3}{3!} + \cdots + \frac{(V_{mn})^{mn}}{(mn)!}$$

(2)

$$| P'_{m,n} - P_{mn} | \leq | P_{m,n} | \cdot \{ e^{V_{m,n}} - 1 \}.$$

From (2), we can see that a sufficient condition for $\lim_{m,n\to\infty} P'_{m,n} = \lim_{m,n\to\infty} P_{m,n}$, provided the latter limit exists, is that $\lim_{m,n\to\infty} V_{m,n} = 0$. Let us assume $\lim_{m,n\to\infty} P_{mn}$ exists, and is different from zero; then $\lim_{m,n\to\infty} u_{m,n} = 0$. Further, assume that $R(u_{m,n}) \geq L > -1$. Then

$$V_{mn} = \sum_{k=1,l=1}^{m,n} \left| \delta_{m,n,k,l} \cdot \frac{u_{k,l}}{1 + u_{k,l}} \right| \leq 1/(1 + L) \sum_{k=1,l=1}^{m,n} | \delta_{m,n,k,l} \cdot u_{k,l} | = 1/(1 + L) \cdot \sigma_{m,n}.$$

If we assume $\sum_{k=1,\,l=1}^{\infty\,\infty} | u_{k,l} |$ converges, then a sufficient condition that $\lim_{m,n\to\infty} V_{m,n}=0$ is that

1) $\lim_{m,n\to\infty} \delta_{m,n,k,l}=0$ for every k and l;

2) $\lim_{m,n\to\infty} \sum_{k=1}^{m} | \delta_{m,n,k,l} | =0$ for each l;

3) $\lim_{m,n\to\infty} \sum_{l=1}^{n} | \delta_{m,n,k,l} | =0$ for each k;

4) $| \delta_{m,n,k,l} | < M$ where M is some constant.

$$\sigma_{m,n}=\sum_{k=1,\,l=1}^{p,q} | \delta_{m,n,k,l}\cdot u_{k,l} | + \sum_{k=1,\,l=q+1}^{p,n} | \delta_{m,n,k,l}\cdot u_{k,l} |$$
$$+ \sum_{k=p+1,\,l=1}^{m,q} | \delta_{m,n,k,l}\cdot u_{k,l} | + \sum_{k=p+1,\,l=q+1}^{m,n} | \delta_{m,n,k,l}\cdot u_{k,l} |.$$

From condition (4), $| \delta_{m,n,k,l} | < K$. Let us choose p and q such that

$$\sum_{k=p+1,\,l=q+1}^{\infty\,\infty} | u_{k,l} | < (\epsilon/4k).$$

Choose M_1 and N_1 so that whenever $m > M_1$, and $n > N_1$,

$$| \delta_{m,n,k,l} | < \epsilon/4\cdot p\cdot qA, \qquad \begin{pmatrix} k=1,2,\cdots,p \\ l=1,2,\cdots,q \end{pmatrix};$$

where $| u_{k,l} | < A$. Choose M_2 and N_2 so that whenever $m > M_2$, and $n > N_2$,

$$\sum_{k=p+1}^{m} | \delta_{m,n,k,l} | < \epsilon/4qA, \qquad (l=1,2,\cdots,q).$$

Choose M_3 and N_3 so that whenever $m > M_3$, and $n > N_3$,

$$\sum_{l=q+1}^{n} | \delta_{m,n,k,l} | < \epsilon/4pA, \qquad (k=1,2,\cdots,p).$$

Let M be the largest of M_1, M_2, and M_3, and let N be the largest of N_1, N_2, and N_3. For any $m > M$, $n > N$,

$$\sigma_{m,n} \leq (\epsilon/4.p.q.A)\cdot p.q.A$$
$$+ (\epsilon/4pA)\cdot p.A + (\epsilon/4.q.A)q.A + (\epsilon/4k)\cdot K=\epsilon.$$

Therefore
$$\lim_{m,n\to\infty} \sigma_{m,n}=0.$$

Expressing these conditions in terms of $a_{m,n,k,l}$, we have the conditions stated in the theorem.

DUKE UNIVERSITY.